A Mathematical Analysis of *Spot It!*

By Deepu Sengupta

Introduction

Spot It! is a game containing a deck of 55 cards with eight symbols printed on each card. Each player looks at a pair of cards and tries to find a symbol that is on both cards. The instructions make a rather extraordinary claim: each pair of cards supposedly has exactly one shared symbol. It seems that the cards must be designed using a special pattern in order to produce this result. What does this pattern look like? Is anything special about the number 55? Could this result have been attained using a different number of symbols on each card?

As strange as it may seem, we can use geometry to answer these questions. In fact, *Spot It!* has almost exactly the same structure as a geometrical object called a *finite projective plane*. Because of my interest in finite projective planes, my mathematics mentor told me about *Spot It!* so that I could explore its relation to projective planes. (She had originally heard of the topic from a presentation by Sue VanHattum at the Math Circle Institute at Notre Dame [4], in which I have participated as a student.)

How can we use geometry to answer questions about a card game? In modern geometry, the words “line,” “point,” and so on are not given any specific definitions. Instead, certain assumptions (or *axioms*) are devised that specify how “lines” and “points” are related to each other. In fact, by using axioms that are different from the axioms of Euclidean geometry (the standard geometry that is taught in grade school), we can create geometries that behave in completely different ways. For example, by changing the axioms, we can create a geometry in which the angles in a triangle add up to less than 180°, one in which parallel lines do not exist, or even one with a finite number of points and lines, without creating any contradictions.

We will take advantage of the flexibility of modern geometry to construct a set of axioms that describes the *Spot It!* game. By refining these axioms somewhat, we will show that *Spot It!* is very similar to a finite projective plane. First, though, we will discuss finite projective planes and the axioms on which they are based.

Finite Projective Planes

The theory of finite projective planes is a topic in projective geometry that mathematicians first investigated in the nineteenth century. Projective planes are based on only three axioms:

1. Any two points are on exactly one line.
2. Any two lines have exactly one point in common.
3. Four points must exist such that no three of them are on the same line.

No further assumptions are made about the behavior of “points” and “lines,” and the words “line” and “point” are not even given any definitions. Thus, lines and points can be anything at all as long as they obey these three axioms.

These axioms can be satisfied using only a finite number of points and lines. For example, consider the arrangement of “points” and “lines” in Figure 1, with seven points (denoted A through G) and seven lines. (This is called the “Fano plane.”) The arrangement is illustrated in Figure 2, in which the dots are points and the connectors are lines. For example, the connector through D, C, and F represents one line. (The circular connector also represents a line.)

Line 1	A	B		D			
Line 2		B	C		E		
Line 3			C	D		F	
Line 4				D	E		G
Line 5	A				E	F	
Line 6		B				F	G
Line 7	A		C				G

Figure 1: The points on each line in the Fano plane.

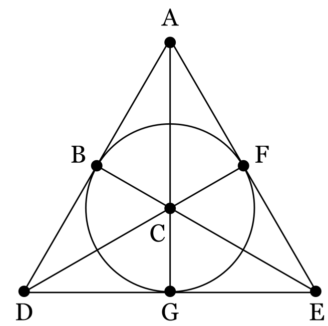

Figure 2: Another depiction of the Fano plane.

Line 1	A	B			E		G						
Line 2		B	C			F		H					
Line 3			C	D			G		I				
Line 4				D	E			H		J			
Line 5					E	F			I		K		
Line 6						F	G			J		L	
Line 7							G	H			K		M
Line 8	A							H	I			L	
Line 9		B							I	J			M
Line 10	A		C							J	K		
Line 11		B		D							K	L	
Line 12			C		E							L	M
Line 13	A			D		F							M

Figure 3: A projective plane with four points on every line.

No matter which two points we choose, only one line contains both points, so this arrangement satisfies Axiom 1. Furthermore, any two lines have exactly one point in common, and no three of the points B, C, D, and G are on a single line, so the arrangement satisfies Axioms 2 and 3 as well.

The example above has three points on each line. Figure 3 shows an example with four points on each line (and 13 points and 13 lines in total).

It has been shown (see Theorem 33 in [2]) that if q is a prime number raised to the power of any natural number, then a finite projective plane must exist with $q + 1$ points on every line and $q + 1$ lines through every point. (The values of q are 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, ...) The question of whether projective planes exist for any other values of q remains unsolved. It can be shown, however, that if a projective plane with $q + 1$ points on every line exists, then the total number of points and lines in the projective plane is $q^2 + q + 1$, regardless of the value of q. (In the example above, $q = 3$ and $q^2 + q + 1 = 13$.)

It follows from the three axioms that in finite projective planes, every line has the same number of points. The proof of this is given in Chapter 6, Problem 8 in [1], and we will summarize it here.

Let l be a line with k points. We will show that every line in this projective plane has k points. Axiom 3 says that all projective planes contain four points such that no three of them are on the same line. This means that the projective plane must contain at least two points Q_1 and Q_2 not on l.

Let $l_1, l_2, ..., l_k$ be the (unique) lines $Q_1P_1, Q_1P_2, ..., Q_1P_k$, respectively, where $P_1, P_2, ..., P_k$ are the k points on l. (See Figure 4.) These lines are distinct, because if, for example, l_1 and l_2 coincided, then Q_1, P_1, and P_2 would be collinear, which contradicts the definition of Q_1.

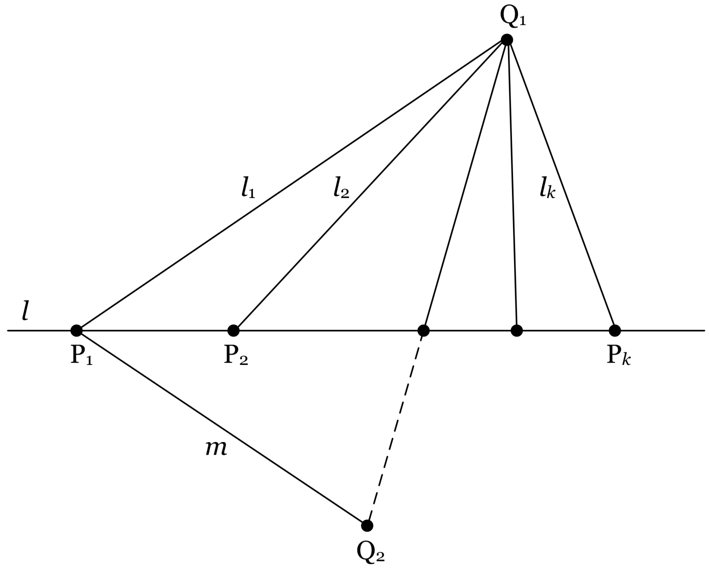

Figure 4: Every line must contain the same number of points.

We first show that a line m exists that is distinct from l and $l_1, l_2, ..., l_k$. Since $l_1, l_2, ..., l_k$ all contain Q_1, at most one of them passes through Q_2 (by Axiom 1). Without loss of generality, we assume that l_1 does not pass through Q_2. Let m be the line Q_2P_1. This line is distinct from l_1 (since l_1 does not pass through Q_2), $l_2, ..., l_k$ (since those lines do not pass through P_1), and l (since l does not contain Q_2).

We now show that the line m, and any line that does not pass through Q_1, contains exactly k points. By Axiom 2, the lines $l_1, l_2, ..., l_k$ each intersect m in exactly one point. By Axiom 1, these k intersection points are distinct, because $l_1, l_2, ..., l_k$ are distinct lines that all pass through Q_1. This means that m contains at least k points. Suppose that m contains j points $R_1, R_2, ..., R_j$. Let $m_1, m_2, ..., m_j$ be the (unique) lines $Q_1R_1, Q_1R_2, ..., Q_1R_j$. Using the same logic as before, we see that $m_1, m_2, ..., m_j$ intersect l in j distinct points, so l contains at least j points. But l contains k points, so k is at least j. By the definition of j, j is at least k, so $j = k$ and m contains exactly k points. These arguments apply not only to m, but to any line that does not pass through Q_1 (except for l itself, which contains k points by definition).

The arguments in the previous paragraph also show that any line that does not pass through Q_2 contains exactly k points. Thus, every line except possibly Q_1Q_2 contains exactly k points. To show that Q_1Q_2 contains k points, let h be the number of points on Q_1Q_2. We can simply repeat the entire argument using the line Q_1Q_2 instead of the line l. This would show that all but one of the lines besides Q_1Q_2 have h points. Thus, as long as at least three lines exist, h must be equal to k. (By Axiom 3 and Axiom 1, three lines must exist.) Thus, Q_1Q_2 and every other line in the projective plane contains exactly k points.

Projective planes have an interesting property called *duality*. In any theorem about projective planes, the words "point" and "line" can be swapped, and the theorem will still be valid. For example, we have just shown that every line in a projective plane has the same number of points. If we replace "point" with "line" and vice versa, we obtain the statement that every point has the same number of lines through it, which is also true. To see why, note that if we swap "point" and "line" in Axiom 1, we obtain Axiom 2. Furthermore, we can prove that if we swap "point" and "line" in Axiom 3, we obtain a statement that is equivalent to Axiom 3.

Spot It! Axioms

We will now develop a system of axioms that we will use to explain *Spot It!* We will see that *Spot It!* bears striking similarities to a finite projective plane.

By the main principle of *Spot It!*, any two cards have exactly one symbol in common. Note that if we call the cards "lines" and the symbols "points," this requirement becomes the second axiom for projective planes. This will be our first *Spot It!* axiom.

(1.) Any two cards ("lines") must have exactly one symbol ("point") in common.

We now need to decide whether we need any more axioms, or whether this axiom alone will be enough. By itself, this axiom allows a variety of "uninteresting" configurations. For example, the axiom can be satisfied even with only one card (that is, one line). The axiom also allows the existence of symbols that are not on any cards. (Since such symbols are not part of the game, we do not need to include them.) To fix this, we add a second axiom:

(2.) Every symbol ("point") is on at least two cards ("lines").

This axiom guarantees that if any symbols exist, then at least two cards must exist. It also prohibits the existence of a symbol that is not on any card.

We also add a third axiom, obtained by swapping the words "point" and "line" in Axiom 2. This eliminates some trivial cases, and it will be useful later when we consider the similarities between "*Spot It!* geometry" and the geometry of finite projective planes.

(3.) Every card ("line") contains at least two symbols ("points").

In *Spot It!*, each card has the same number of symbols on it. The three axioms we have so far do not guarantee this. For example, consider the arrangement of points (symbols) and lines (cards) in Figure 5. All three of the above axioms hold, but the line BCD has three points and the other lines have only two points.

We would like every card to have the same number of symbols, so we add a fourth axiom:

(4.) All cards ("lines") contain the same number of symbols ("points").

The four axioms we have so far allow a configuration with no points or lines at all. To eliminate this possibility, we add a fifth axiom:

(5.) At least one symbol ("point") exists.

By Axiom 2, it follows that at least one card (in fact, at least two cards) must exist.

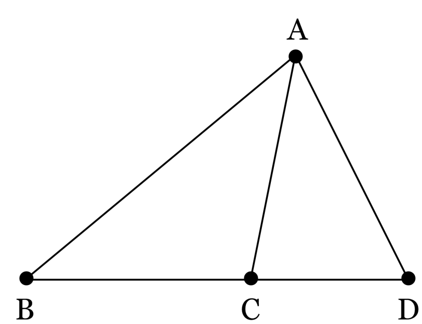

Figure 5: An arrangement in which the lines do not all have the same number of points.

The Structure of *Spot It!*

Now that we have created a system of axioms, we can investigate the pattern of cards and symbols in the *Spot It!* game.

We first show that every finite projective plane has a configuration similar to that of the cards and symbols in the *Spot It!* game. To do this, we show that the *Spot It!* axioms follow from the axioms for projective planes. The first *Spot It!* axiom is exactly equivalent to the second axiom for projective planes. Furthermore, in every projective plane, each line has at least three points, and each point is on at least three lines (we will prove this later). Thus, *Spot It!* axioms 2 and 3 hold for all projective planes. Also, as proved earlier, all lines in a projective plane have the same number of points, so Axiom 4 also holds. Finally, Axiom 5 follows directly from the third axiom for projective planes. Therefore, we can conclude that the axioms for projective planes imply the *Spot It!* axioms.

This means that if q is a natural-number power of a prime number, we can create a *Spot It!* game with $q+1$ symbols on each card, $q+1$ cards with each symbol, q^2+q+1 cards (in total), and q^2+q+1 symbols. The values of $q+1$ are 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 18, 20, ...

Most versions of the *Spot It!* game have eight symbols on each card, so it seems that they could be equivalent to the $q = 7$ projective plane. In that case, we would expect the total number of cards to be $7^2 + 7 + 1 = 57$. But the versions of *Spot It!* with eight symbols on each card contain only 55 cards. So even though every projective plane satisfies the *Spot It!* axioms, not every configuration satisfying the *Spot It!* axioms corresponds to a projective plane.

What, specifically, is the pattern with 55 cards? As discussed in [3], it is simply the $q = 7$ projective plane with two lines (i.e., cards) removed. To see how this makes sense, let us return for a moment to the projective plane shown in Figure 3. Suppose that we simply remove two of the lines (for example, lines 12 and 13). We can see that all of the *Spot It!* axioms still hold, even though the configuration is no longer a projective plane. In fact, as long as we do not remove *all* of the lines, Axioms 1, 3, 4, and 5 hold no matter how many lines we remove. Thus, we can remove as many lines as we want as long as Axiom 2 is still satisfied.

Similarities between *Spot It!* and Projective Planes

By modifying the *Spot It!* axioms somewhat, we obtain a system of axioms that is exactly equivalent to the axioms for projective planes. We now discuss these modifications.

The five *Spot It!* axioms do not result in a *dual* geometry (i.e., a geometry with the property of duality described above). For example, if we swap "symbol" and "card" in *Spot It!* axiom 1, we obtain the statement that any two symbols have exactly one card in common (that is, any two symbols are on exactly one card). This is equivalent to the first of the axioms for projective planes, but it is not necessarily true for *Spot It!*, because if we remove a card (line) from a finite projective plane as described above, we remove the unique card containing some particular pair of symbols, and so it is no longer true that any two symbols are on exactly one card.

Since projective planes are dual, we will need to adjust the *Spot It!* axioms to make them equivalent to the axioms for projective planes. We simply add one more axiom, which we obtain by swapping "symbol" and "card" in Axiom 1:

(6.) Any two symbols ("points") are on exactly one card ("line"). (This is equivalent to the first of the axioms for projective planes.)

We do not need to add the "dual" of Axiom 4 (i.e., the statement obtained by swapping "symbol" and "card" in Axiom 4), because we will now show that the dual of Axiom 4 (i.e., "all symbols are on the same

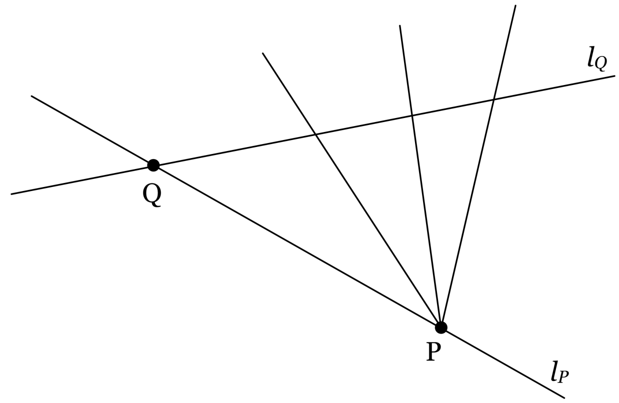

Figure 6: A proof that all points are on the same number of lines.

number of cards") follows from Axiom 4 itself. We will use the words "point" and "line," and prove that all points are on the same number of lines.

Let k be the number of points on each line (i.e., the number of symbols on each card), and let P be any point. (See Figure 6.) Any line l_P that passes through P must contain at least one other point Q distinct from P (Axiom 3), and this point must be on at least one line l_Q distinct from l_P (Axiom 2). The line l_Q cannot pass through P (Axiom 1). Since l_Q contains exactly k points, exactly k lines exist joining P to the k points of l_Q (Axiom 6). Every line through P must intersect l_Q in a unique point, because two lines through P and intersecting l_Q in the same point would violate Axiom 1. Thus, P has exactly k lines through it. (This proof is similar to ideas in Chapter 6, Problem 8 in [1].)

These six axioms are not exactly equivalent to the axioms for projective planes, because they allow the configuration shown in Figure 7. This is not a projective plane, because it violates the third axiom for projective planes (which says that four points must exist such that no three of them are on a single line).

To eliminate this possibility, we will modify two of the axioms very slightly:

(2.) Every symbol ("point") is on at least *three* cards ("lines").

(3.) Every card ("line") contains at least *three* symbols ("points").

These changes rule out the possibility above, since each line must now have at least three points.

Here, then, are the modified *Spot It!* axioms. (The words "card" and "symbol" can be replaced with "line" and "point," respectively.)

1. Any two cards have exactly one symbol in common. (Equivalent to projective-plane axiom 2)
2. Every symbol is on at least three cards.
3. Every card contains at least three symbols.

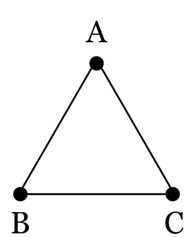

Figure 7: Not a projective plane.

4. All cards contain the same number of symbols.
5. At least one symbol exists.
6. Any two symbols are on exactly one card. (Equivalent to projective-plane axiom 1)

We now show that these six *Spot It!* axioms are exactly equivalent to the three axioms for projective planes. *Spot It!* axioms 6 and 1 are equivalent to projective-plane axioms 1 and 2, respectively. We show first that Axiom 3 for projective planes (which specifies that four points exist such that no three are on the same line) follows from the six *Spot It!* axioms.

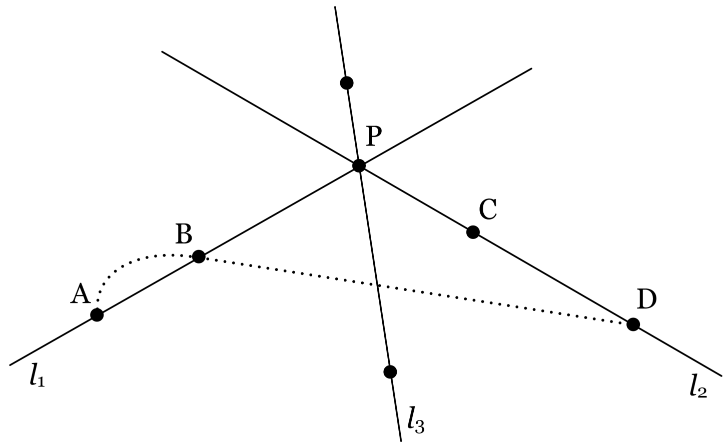

Figure 8: Axiom 3 for projective planes follows from the *Spot It!* axioms.

By Axiom 5, at least one point P must exist, and by Axiom 2, at least three lines l_1, l_2, and l_3 must pass through P. (See Figure 8.) By Axiom 3, each of these three lines must contain at least two points other than P. Let A and B be two points on l_1 distinct from P, and let C and D be two points on l_2 distinct from P.

Any set of three points in $\{A, B, C, D\}$ contains two points on either l_1 or l_2, so, by Axiom 6, the three points cannot be on a line. (For example, consider the set $\{A, B, D\}$. The points A and B are both on l_1. Since D is not on l_1, ABD cannot be a line, because otherwise, the lines ABD and l_1 would both join A and B, which violates Axiom 6.) Therefore, the points A, B, C, and D satisfy the third axiom for projective planes.

This shows that the axioms for projective planes follow from the six *Spot It!* axioms. Note that we did not refer to the fourth *Spot It!* axiom in this proof. Therefore, the axioms for projective planes follow from the other five axioms. But the fourth *Spot It!* axiom (i.e., "all lines have the same number of points") follows from the axioms for projective planes. This means that making the above modifications to the axioms caused Axiom 4 to become redundant, since it now follows from the other five axioms. Additionally, the proof above would still be valid even if the line l_3 (see Figure 8) did not exist, so when we modified *Spot It!* axioms 2 and 3 above, it was not necessary to modify Axiom 2 so that it guaranteed the existence of at least three lines (instead of two lines) through any given point.

We will now show that the (modified) *Spot It!* axioms follow from the axioms for projective planes. The fifth *Spot It!* axiom follows directly from Axiom 3 for projective planes. We showed earlier that every line in a projective plane must contain the same number of points, so the fourth *Spot It!* axiom follows from the axioms for projective planes as well. To obtain the second and third *Spot It!* axioms from the axioms for projective planes, consider the four points A, B, C, and D guaranteed by Axiom 3 for projective planes, no three of which are collinear. (See Figure 9.) Three distinct lines must exist joining A to B, to C, and to D (because of Axiom 1 for projective planes and the fact that no three of the four points are collinear). Furthermore, the lines AB and CD do not intersect at A, B, C, or D (since no three of those points are

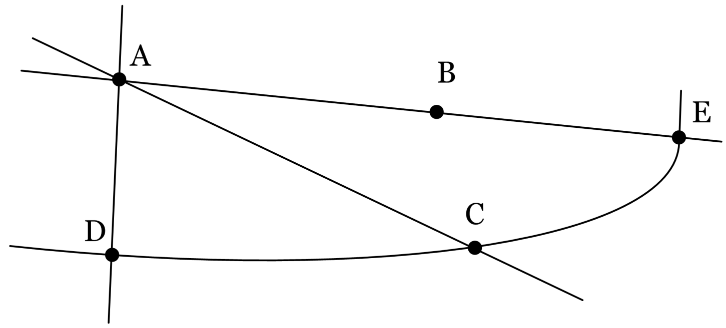

Figure 9: The second and third *Spot It!* axioms follow from the axioms for projective planes.

collinear), so they must intersect at some other point E on AB. Thus, the point A has at least three lines through it, and the line AB has at least three points on it. By the fourth *Spot It!* axiom, it follows that all points are on at least three lines, and all lines have at least three points.

This shows that the modified *Spot It!* axioms are equivalent to the axioms for projective planes.

We can see from all of this that geometry can be used to model situations that do not seem at all geometrical. If we limit ourselves to thinking of geometry as the study of "points" and "lines" drawn on a piece of paper, we can only take advantage of a small part of its usefulness. It is also surprising that such a simple-sounding card game is so closely related to advanced geometrical concepts.

Further Reading

In [5], [6], and [7], Sue VanHattum discusses *Spot It!* and her experience with using it as a topic in schools and Math Circles.

In [3], Michael Kleber discusses the missing two cards in the *Spot It!* game. A comment on Kleber's post by Guillaume Gille-Naves, a former member of the development team for the French version of *Spot It!*, explains that the game has 55 cards (instead of 57) for marketing purposes and to ensure that the rules of the game could be printed on extra cards without exceeding the manufacture limitation of 60 cards.

Finally, an article [8] by Burkard Polster (published in the magazine *Math Horizons* shortly after I wrote this paper) discusses the mathematics behind *Spot It!*, including the game's similarities to projective planes and other mathematical objects.

References

[1] Sved, Marta. *Journey into Geometries.* The Mathematical Association of America, Washington, D.C., 1991.

[2] Kåhrström, Johan. *On Projective Planes.* Retrieved from http://kahrstrom.com/mathematics/documents/OnProjectivePlanes.pdf on April 23, 2015.

[3] Kleber, Michael. Post on Google+ about the missing two cards in *Spot It!* Retrieved from https://plus.google.com/+MichaelKleber/posts/Fa9Znut5JmK on January 31, 2016.

[4] Riverbend Community Math Center. *Math Circle Institute.* Retrieved from http://riverbendmath.org/math_circle_training_institute/ on January 31, 2016.

[5] VanHattum, Sue. *Math Adventures: Thinking about Spot it, and Learning Python.* Retrieved from http://mathmamawrites.blogspot.com/2012/01/math-adventures-thinking-about-spot-it.html on January 31, 2016.

[6] VanHattum, Sue. *The Oakland Math Circle does Spot It.* Retrieved from http://mathmamawrites.blogspot.com/2012/03/oakland-math-circle-does-spot-it.html on January 31, 2016.

[7] VanHattum, Sue. *Spot It Circle Today.* Retrieved from http://mathmamawrites.blogspot.com/2012/06/spot-it-circle-today.html on January 31, 2016.

[8] Polster, Burkard. *The Intersection Game.* Published in *Math Horizons.* Retrieved from http://www.maa.org/sites/default/files/pdf/horizons/PolsterApril2015.pdf on January 31, 2016.