Math Teachers' Circle Lesson:
Counting dots on cubes and tetrahedra

Andrew Gillette - University of Arizona

May 30, 2017

Abstract. How can you put dots on a cube or a tetrahedron in a symmetric way? We will look at a special deck of playing cards that shows many ways to do this and learn about the "magic square" type patterns that are hidden in these arrangements.

Note to the instructor. This activity is based on the Periodic Table of Finite Elements, viewable online at https://femtable.org. Even if you know absolutely nothing about this table or about finite elements, it is still possible to do the following activity as it only uses some basic geometry and counting techniques.

The activity requires a special deck of playing cards, shown at left. If you know someone who studies finite elements, they might have a deck. If not, you can print out copies of the cards using the pdf file: https://fenicsproject.org/pub/graphics/fenics-femtable-cards.pdf The concept and scientific content of the cards were developed by Douglas N. Arnold (University of Minnesota) and Anders Logg (Chalmers University of Technology).

Materials.
- 1 deck of "Periodic Table of the Finite Elements" playing cards, or printout of pdf file (see comment next to above photo)
- Rubik's cube and square-base pyramid (optional, but helpful visual aid)
- Board space for class and scratch paper for participants

Preparation. Before teachers arrive, draw the following tables on the board:

$\mathcal{P}_1^-\Lambda^k$	0	1	2	3
boundary				
all				
interior				

$\mathcal{Q}_1^-\Lambda^k$	0	1	2	3
boundary				
all				
interior				

Introduction - Overview and Discussion

There are basic physical phenomena such as the diffusion of heat through a room or particles flowing in your bloodstream that can be *described* by mathematics but require computer simulations to better understand. One way mathematicians help this process is by designing simulation methods that use computing power very efficiently.

A nice way to draw people into this idea is to show the following two videos:
Blade out test on a turbofan engine:
https://www.youtube.com/watch?v=wcALjMJbAvU
Simulation of Fan Blade containment test:
https://www.youtube.com/watch?v=8IB6kO4wHUk
The simulated test is obviously much more cost effective (you don't have to physically build and destroy a jet engine), but the simulation might be computationally expensive, or inaccurate.

Draw a rectangle on a central board. If we want to simulate something on this region we can break it up into many small pieces and do some computations over each piece. Illustrate two ways to break up the rectangle: into squares, or into triangles. Ask the class: what would the 3D analog of squares and triangles be? (Answer: cubes and tetrahedra).

We are going to look at a deck of cards, each of which has either a cube or a tetrahedron on it. These "elements" can be glued together to make 3D regions and do computer simulations in 3D. For this lesson, we are interested in how many computations such a procedure requires.

Drawn on the shape on each card are some dots or arrows, each of which represents "1 computation" that the computer would have to do. This has two consequences:

$$\begin{aligned} &\text{More dots} \implies \text{More computations} \\ &\text{More dots} \implies \text{More accurate simulation} \end{aligned}$$

Our goal is to find some patterns in how the dots appear on the cards and then, later, to see if we can find some cards that are not in the deck.

Activity 1 - Counting dots on $\mathcal{P}_r^-$ and $\mathcal{Q}_r^-$.

Show the four $\mathcal{P}_1^-\Lambda^k$ cards (numbers 2, 3, 4, 5 in the $\mathcal{P}-$ suit; first 4 pages of the pdf file), or draw them on the board. Fill out the $\mathcal{P}_1^-\Lambda^k$ table as a class (correct answers shown below).

- Explain how k can be found on the cards (superscript of Λ) or by color ($k = 0, 1, 2, 3$ corresponds to green, red, yellow, blue, respectively).
- Explain what is meant by the boundary (any dot or arrow on a vertex, edge or face) versus the interior (any dot or arrow in the middle of the shape).
- Explain that 'all' is all dots (boundary + interior) and that number is shown on the cards. If there still seems to be confusion, show the four $\mathcal{Q}_1^-\Lambda^k$ cards (numbers 2, 3, 4, 5 in the $\mathcal{Q}-$ suit) and do the same exercise as a class.

Now put teachers into groups of 2-3 and give each group a stack of cards as follows:

Stack 1: four $\mathcal{P}_2^- \Lambda^k$ cards (numbers 6, 7, 8, 9 in the $\mathcal{P}-$ suit)
Stack 2: four $\mathcal{P}_3^- \Lambda^k$ cards (numbers 10, J, Q, K in the $\mathcal{P}-$ suit)
Stack 3: four $\mathcal{Q}_2^- \Lambda^k$ cards (numbers 6, 7, 8, 9 in the $\mathcal{Q}-$ suit)
Stack 4: four $\mathcal{Q}_3^- \Lambda^k$ cards (numbers 10, J, Q, K in the $\mathcal{Q}-$ suit)

The groups should create tables, like the ones for $\mathcal{P}_1^- \Lambda^k$ and $\mathcal{Q}_1^- \Lambda^k$ for each stack. Once a group has finished with a stack, they can trade with another group. Correct answers for all stacks, plus the intro examples appear later in this document.

Questions to prod discussion:

- What is the relationship among the numbers in each column? (boundary + interior = all)
- Where do dots/arrows appear on the geometry for $k = 3$? $k = 2$? $k = 1$? $k = 0$? (Answer: only on $\geq k$ dimensional sub-faces)
- Describe the symmetry of the arrangement of the dots/arrows (Answer: same number appear on each sub-geometry type, e.g. 2 arrows on every edge)

As groups finish, have them write the results on boards around the classroom. Make sure all groups agree on the results that others have written up. For groups that finish quickly, give them a challenge question such as

- Can you find a pattern across the rows of each table? (Hint: rewrite equation as: boundary - all + interior = 0)
- Can you fill out some or part of the table for $\mathcal{P}_4^- \Lambda^k$ or $\mathcal{Q}_4^- \Lambda^k$? (There are no cards for these!)

Answer Key for Activity 1

The columns are labeled by k value (0, 1, 2, 3), given by the superscript for Λ on each card. The "$\pm \sum$" column will be explained in Activity 2 and should be left blank for Activity 1.

$\mathcal{P}_1^- \Lambda^k$	0	1	2	3	$\pm \sum$
boundary	4	6	4	0	2
all	4	6	4	1	1
interior	0	0	0	1	-1

$\mathcal{Q}_1^- \Lambda^k$	0	1	2	3	$\pm \sum$
boundary	8	12	6	0	2
all	8	12	6	1	1
interior	0	0	0	1	-1

$\mathcal{P}_2^- \Lambda^k$	0	1	2	3	$\pm \sum$
boundary	10	20	12	0	2
all	10	20	15	4	1
interior	0	0	3	4	-1

$\mathcal{Q}_2^- \Lambda^k$	0	1	2	3	$\pm \sum$
boundary	26	48	24	0	2
all	27	54	36	8	1
interior	1	6	12	8	-1

$\mathcal{P}_3^- \Lambda^k$	0	1	2	3	$\pm \sum$
boundary	20	42	24	0	2
all	20	45	36	10	1
interior	0	3	12	10	-1

$\mathcal{Q}_3^- \Lambda^k$	0	1	2	3	$\pm \sum$
boundary	48	100	54	0	2
all	64	144	108	27	1
interior	16	44	54	27	-1

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Break

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Activity 2: Deriving dots for a new family on cubes

Looking at the elements already computed, let's discuss again what the numbers mean. The subscript number (e.g. 1, 2, 3, on $\mathcal{Q}_1^-$, $\mathcal{Q}_2^-$, $\mathcal{Q}_3^-$), represents the order of accuracy of the method. For instance, a $\mathcal{Q}_2^-$ method is accurate to "quadratic order" while a $\mathcal{Q}_1^-$ method is accurate to "linear order". The k number (the column headers) represent the dimension of thing to be modeled (formally, this is the differential form order). For instance, an electrical current travels along a wire and is a 1D phenomenon ($k = 1$), while a magnetic field occurs over a flat region (think of a magnet) and is a 2D phenomenon ($k = 2$).

Note that the $\mathcal{Q}_3^-$ elements involve a lot of computations - for instance, $\mathcal{Q}_3^-\Lambda^1$ has 144 computations per element! This raises the question: can we find a new group of elements, also on cubes, that satisfies the same pattern but uses fewer computations?

First, we must find a pattern in the rows. Let groups think about this for a while. When they need a hint, suggest doing a sum across each row but with alternating signs. Starting with + for $k = 0$, they will find a value of 2 for the boundary row, 1 for the "all" row, and -1 for the interior row. This can easily be confirmed for all the computed tables on the boards; it may be helpful to write a new column $\pm\sum$ on each table to emphasize this point (as shown in the answer key).

$\mathcal{S}_1^-\Lambda^k$	0	1	2	3	$\pm\sum$
boundary	8			0	
all	8			1	
interior	0			1	

Present the above table and ask the teachers to use what they have learned to fill in the rest. Explain that 8 is the minimum for $k = 0$ (one dot per vertex) and 1 is the minimum for $k = 3$ (one dot for the interior). The strategy the teachers should use is to first fill in the $\pm\sum$ column with 2, 1, -1, then look for patterns in the elements and make some guesses to deduce the table. Then draw similar tables with only the 0 and 3 columns filled in for $\mathcal{S}_2^-\Lambda^k$, $\mathcal{S}_3^-\Lambda^k$, and $\mathcal{S}_4^-\Lambda^k$. [1]

Questions to prod discussion:

- What can the boundary count be for $k = 2$? (must be a multiple of 6)
- What can the boundary count be for $k = 1$? (must be of the form $12e + 6f$)
- How do these elements compare entry by entry to the $\mathcal{Q}^-$ elements?

[1] The spaces in Activity 2 are *not* the $\mathcal{S}_r\Lambda^k$ spaces from the playing cards; they are different, as the superscript "−" indicates. The discovery of these spaces is explained in the paper Gillette, Kloefkorn, "Trimmed serendipity finite element differential forms." arXiv:1607.00571 (2016).

Answer Key for Activity 2

$\mathcal{S}_1^-\Lambda^k$	0	1	2	3	$\pm\sum$
boundary	8	12	6	0	2
all	8	12	6	1	1
interior	0	0	0	1	-1

$\mathcal{S}_2^-\Lambda^k$	0	1	2	3	$\pm\sum$
boundary	20	36	18	0	2
all	20	36	21	4	1
interior	0	0	3	4	-1

$\mathcal{S}_3^-\Lambda^k$	0	1	2	3	$\pm\sum$
boundary	32	66	36	0	2
all	32	66	45	10	1
interior	0	0	9	10	-1

$\mathcal{S}_4^-\Lambda^k$	0	1	2	3	$\pm\sum$
boundary	50	108	60	0	2
all	50	111	82	20	1
interior	0	3	22	20	-1

Closing remarks

Show Periodic Table of Finite Elements online: https://femtable.org/

Discuss how the playing cards relate to the organization of the table.

Perhaps show additional of simulations carried out using finite elements.

For a challenge, or as an additional activity, participants can seek out similar patterns using the other two families ($\mathcal{P}_r\Lambda^k$ and $\mathcal{S}_r\Lambda^k$), although for those families, you have to go down by 1 in r each time you go up by one in k, which requires you to deduce what the cards would look like for $r > 3$. An example is shown here:

$\mathcal{P}_{4-k}\Lambda^k$	0	1	2	3	$\pm\sum$
boundary	34	56	24	0	2
all	35	60	30	4	1
interior	1	4	6	4	-1

Similar tables could be made for $\mathcal{P}_{5-k}\Lambda^k$, or $\mathcal{S}_{4-k}\Lambda^k$, etc.