Rutgers Math Teachers' Circle at Toms River – Feb 18 2015

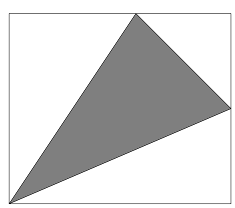
(i) **Geoboards.** We didn't have time to demonstrate the online tool, but it is free and easily available. So there are activities one could do with polygons, without having geoboards available in the classroom. (Just google on-line geoboard, or try http://www.mathlearningcenter.org/web-apps/geoboard/)

(ii) **Equilateral triangles on geoboard.**

First, let's clarify we're referring to geoboards whose pegs are arranged in "square" fashion, not "equilateral triangle" fashion. Also, we're trying to answer the question in the context of an infinitely big geoboard, so that we're not limited in the types and lengths of segments we can create. Lastly, we agreed that the question is referring to triangles whose vertices are all pegs.

With these in mind, the answer is…. (Drumroll)

Equilateral triangles cannot be made on a geoboard. Here is one way of seeing this. First, prove that any triangle drawn on a geoboard must have a rational area. To that end, you can draw vertical lines through the left-most and right-most vertices of the triangle, and horizontal lines through the highest and lowest vertices. They will intersect to form a rectangle whose sides are of integer length, so its area is an integer. The area of the triangle inside is the area of this rectangle minus the sum of the areas of 3 little triangles that are inside the rectangle but outside the initial triangle. Integer minus rational gives rational.



Now, the formula for the area of an equilateral triangle is $side^2 * \sqrt{3}/4$. Because the coordinates of the triangle's vertices are integers, side^2 is an integer as well (as a consequence of the Pythagorean theorem applied in a triangle with integer leg lengths). So, $side^2 * \sqrt{3}/4$ is irrational (as a rational times an irrational). This contradicts the conclusion of the paragraph above, which was that the area of the equilateral triangle must be rational. Therefore this triangle does not exist.

Patty (I think) asked an interesting related question: what if we are allowed to use multiple elastic bands to create the triangle, and not all of its vertices have to be pegs (for example, one of them can be at the intersection of two elastic bands)? Can we create an equilateral triangle that way? I haven't thought about that yet so we should think together.

(iii) Theorem on area of polygons. This theorem is known as Pick's theorem. A key issue that came up is data collection: this was a key feature in how quickly people discovered the formula, which is:

$$\text{Area} = i + b/2 – 1$$

(where i is the number of pegs inside the polygon and b the number of pegs on the boundary of the polygon).

I think the key point here is that one has to mess around a little bit first, in order to set up a data collection scheme, and then go back and do things more systematically; the data collection scheme has to be “designed” based on some experience that comes from playing around with the given situation, even if at first one may not have a clear idea how to organize the data or what kind of pattern to look for.

Another issue that came up is what kind of formula can one expect? Given that it’s about area, should it have something squared? Could one expect a formula just expressing the area in terms of the number of lattice points? why or why not? etc. Chelsea noted that what was helpful for her was to think about the formula as something of the form A = …., and that this gave her a way to organize the data by keeping one of the “peg” variables constant and varying the other while noticing what happens to the area.

We didn’t have time to talk much about how to show that the formula works for all polygons. A related but different question is, why would there be a connection between area and number of pegs? Conceptually, what connects them? And why did someone decide to look for a relationship between area and number of pegs, in the first place?

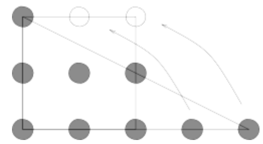
The following article was written as a by-product of a project conducted by middle and high-schoolers, under the supervision of a mathematician. I think you’ll like its main idea, which is accessible at the K-12 level. It also provides an explanation of how to reconcile the units on the two sides of the equation (square units on the left side, number of pegs on the other???). The gist of the article is presented in the first 1-2 pages; the rest deal with the details of more general cases, as well as with generalizations of Pick’s theorem.

http://www.jamestanton.com/wp-content/uploads/2009/04/picks_theorem_focus_web-version.pdf

Another (more formal) argument is to look at the quantity n(P) given by “number of points inside or on the boundary minus half the points on the boundary minus one” and show that it satisfies an “additivity principle”: whenever P is divided up into P_1 and P_2 then $n(P) = n(P_1) + n(P_2)$. For triangles of area one-half, one shows that the n(P) is 1/2, which is the same as the area. But any geoboard polygon can be divided into geoboard triangles of area one-half (challenge)!

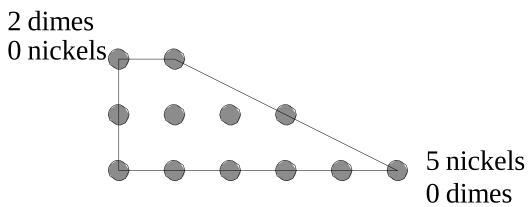
(iv) **Number of ways of making change for a dollar.** This is a problem that a lot of fourth graders get sent home with, and we tried to make the connection with geometry here. Here are some of the main points of the solution.

The number of ways of making change for a multiple of ten cents using pennies, nickels and dimes is the number of pegs in a geoboard triangle whose hypotenuse has slope −1/2. The number of such points is always a square (you can rearrange the points in the triangle into the points in a square, by taking the points on the “pointy half” and moving them up top”):



So for example, the number of ways of making change for 50 cents is 36, and for a dollar, 121, without using quarters; for making change for 10n cents there are $(n+1)^2$ ways.

The number of ways of making change for a number like 25, 75 using pennies, nickels and dimes is the number of lattice points inside a trapezoid. (For 25 cents, this is the trapezoid with vertices (0, 0), (0, 2), (1, 2), (5, 0).)



Cutting off the left bit of the trapezoid one gets a triangle. The number of ways of making change for 25 cents is 3 + 9 = 12 ways, and the number of ways of making change for 75 cents is 8 + 64 = 72 ways. (It’s easier to see this if you draw the trapezoid here, with vertices (0, 0), (0, 7), (1, 7), (15, 0).)

To make change for a dollar, you can either use 0,1,2,3, or 4 quarters. The leftover amounts are 0,25,50,75,100 cents which you have to make change for using pennies, nickels and dimes. So in total there are 1 + 12 + 36+ 72 + 121 = 242 ways of making change for a dollar (trapezoid + triangle + trapezoid + triangle) without using half-dollar or dollar coins.

If you allow one half-dollar, then you get another 1 + 12 + 36 = 49 ways, and one can also use two-half dollars or just a dollar coin(does this count as making change?) for a total of 293 ways.

There are a bunch of discussions of this on the web, see for example http://www.maa.org/frank-morgans-math-chat-293-ways-to-make-change-for-a-dollar.