Regular, Semi-regular, and Escher-like Tilings

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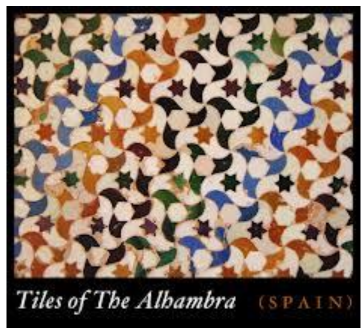
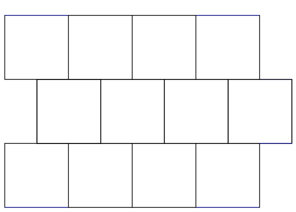
"*... [Mathematicians] have opened the gate leading to an extensive domain, but they have not entered this domain themselves. By their very nature they are more interested in the way in which the gate is opened than in the garden lying behind it."* –M. C. Escher

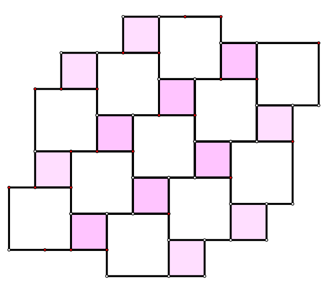
Escher's interest in tilings began in earnest in the 1930's following a visit to the Alhambra, a castle in Spain built in the 9th century and converted to a palace by the Moors in the 14th century. There he sketched many different examples of intricate tilings, such as the ones below:



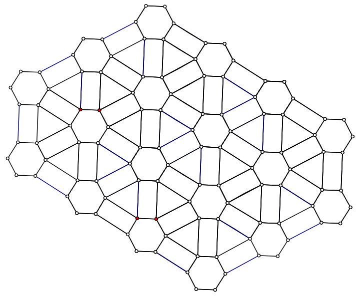
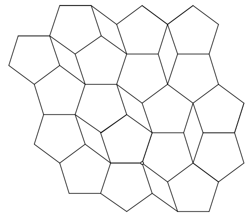
A tiling, or tessellation, of the plane is a covering without gaps or overlaps, by congruent copies of one or more shapes. Although any shape is allowed, we will restrict ourselves initially to tilings by polygons. Let's start with some terminology:

The **order of a tiling** refers to the number of non-congruent figures used in the tiling. What is the order of each tiling below?





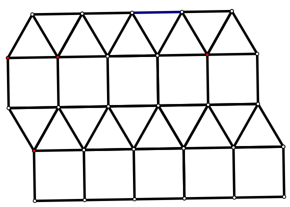
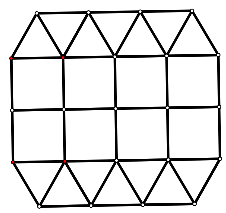
A tiling is called **edge-to-edge**, if neighboring figures meet along a complete edge of each figure. The two square tilings above are not edge-to-edge, but the following tilings are. What are the orders of these next two tilings?

A tiling is called **regular** if it is an order 1, edge-to-edge tiling by a regular polygon.

PROBLEM: Find all regular tilings (up to size transformation). How do you know that you have found them all?

A tiling is called **semi-regular** if is an edge-to-edge tiling by regular polygons, and the pattern of polygons surrounding every vertex is identical. For example, the tiling on the left below is semi-regular, but the one on the right is not. Are any of the previous tilings semi-regular?

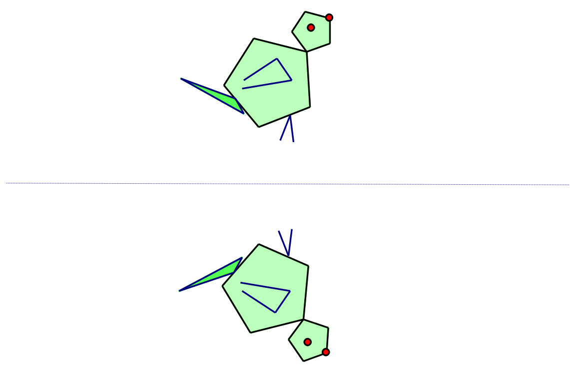
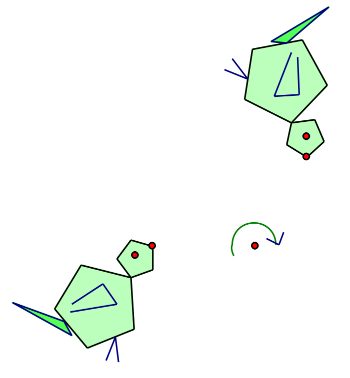
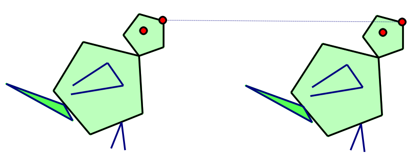
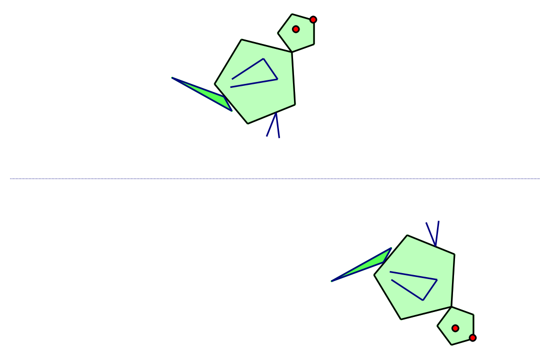
 

Semi-regular tilings are denoted by their vertex patterns. For example, the semi-regular tiling above is denoted by 3.3.3.4.4 because as you travel around each vertex you will see three triangles, followed by two squares. Note that 3.3.4.4.3 would give the same tiling, as would 3.4.4.3.3, etc.. However 3.3.4.3.4 is not the same. Do you see why not?

Question: Does the vertex pattern 3.3.4.3.4 represent a semi-regular tiling? Does 5.5.10?

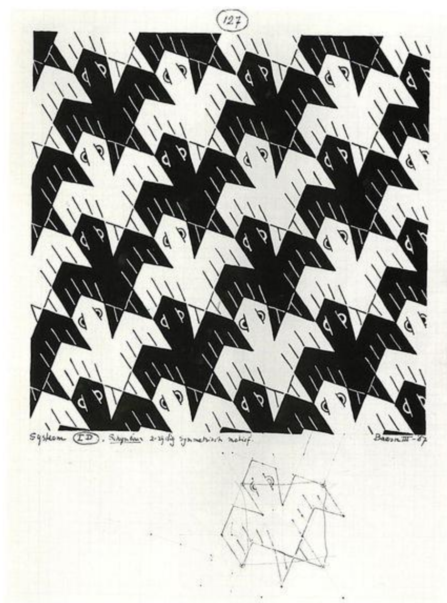
MTCA Fall 2013 Challenge: Find all semi-regular tilings (up to size transformation). Explain clearly how you know you have found them all.

Escher-like tilings: Escher's beautiful and creative tilings by animals and people are, in many cases, obtained by modifying a very simple tessellation of the plane. The key idea is to modify each tile in such a way that the modified tile still tessellates. This is done by altering the tiles in a way that reflects the symmetries of the desired tiling. We will consider symmetries with respect to the four motions illustrated below:

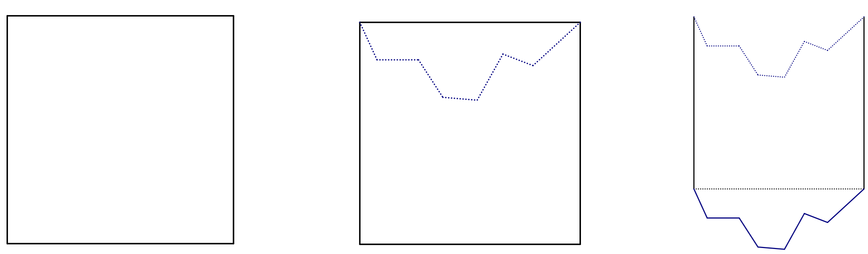
Reflection across a line	Rotation about a point
	
Translation by a vector	Glide reflection
	

Examine each of the regular tilings you have created and determine what kinds of symmetries it exhibits. All regular and semi-regular tilings have translational symmetry in at least two different directions. Such tilings are called **wallpaper patterns**. Any wallpaper pattern can be used as the basis for an Escher-like tiling by exploiting the translational symmetry.

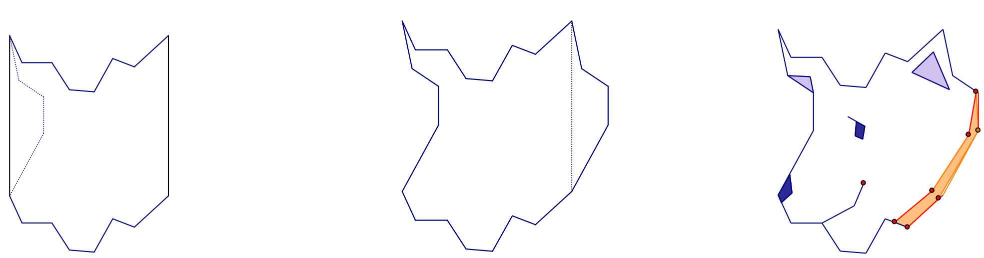
Let’s consider how we can use translational symmetry to produce an Escher-like tiling. Look at Escher’s tiling by birds. The foundation for this tessellations is a regular tiling by squares, as you can see by the figure at the bottom of the sketch. Notice that the modification of the top edge of the square is translated to the bottom edge. Likewise, the modification on the left edge is translated to the right edge. This guarantees that the modified tile will still have horizontal and vertical translational symmetry.



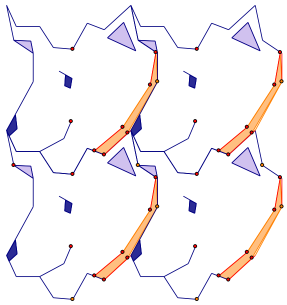
Now your turn: Start with a square cut out from an index card, approximately 2 inches in length, and cut a piece off of the top edge. Tape the cutout piece to the bottom edge of the square. You have translated the modification of the top edge to the bottom.



Do the same on the sides: cut a piece off of the left side and tape it to the right side and decorate.

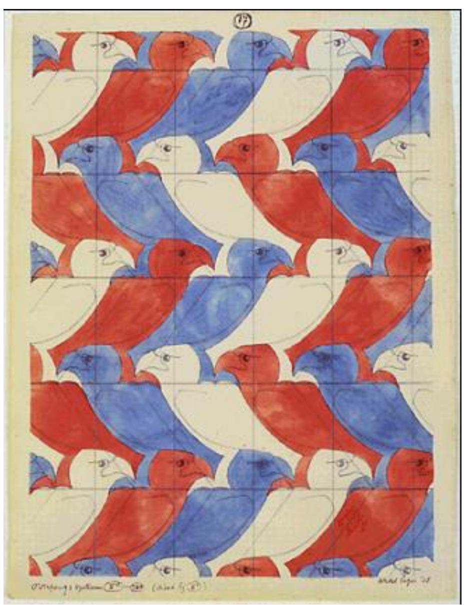


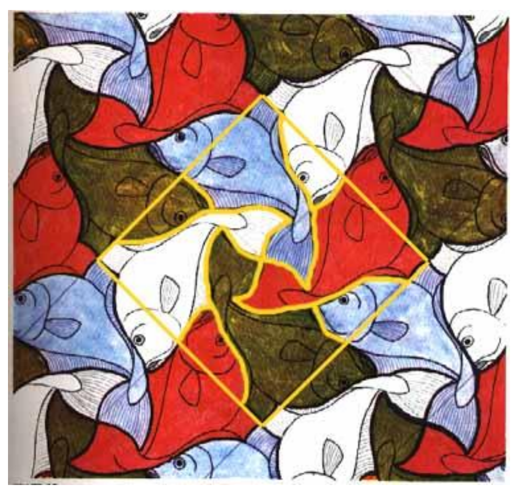
Now you are ready to tile: Trace around the template you have made and add details.



Getting fancier:

Consider the two Escher tessellations below. What kind of symmetries do these tilings have (ignoring the coloring)? How did Escher modify the edges of the rectangle to produce the tiling on the left? How did he modify the sides of the square to produce the one on the right?



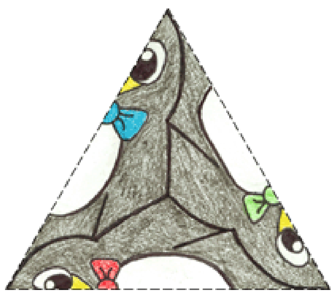


Now your turn #2: Cut a square from an index card and modify it to produce an Escher-like tiling with the same symmetries as one of the tilings above.

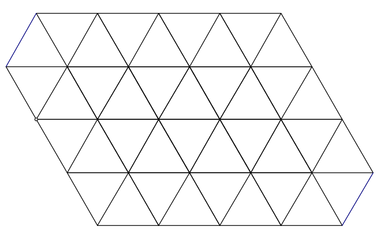
Extension: Try different combinations of the ideas contained in these two tessellations to find a tessellation based on a square or rectangular tiling that has symmetries that are different from these in some way.

Escher-like tilings based on regular tilings by equilateral triangles:

A nice idea for using symmetry in a simple way to produce an interesting tiling by equilateral triangles comes from Jill Britton's web page, "Escher in the Classroom". (http://britton.disted.camosun.bc.ca/jbescher.htm) The idea is to use reflectional symmetry across edges to decorate each triangle without modifying the edges. Here is are two examples taken from the "Escher in the Classroom" web site:



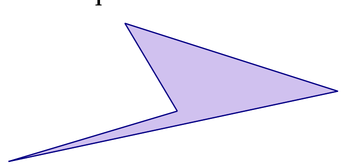
Suppose we did want to modify the edges of the triangle. What other kinds of symmetry of the regular tiling by triangles could we exploit? How could the triangular tile be modified to produce an Escher-like tiling?



Now your turn #3: Start with an equilateral triangle cut from an index card. Modify all three edges in some way that will result in a figure that will tile the plane.

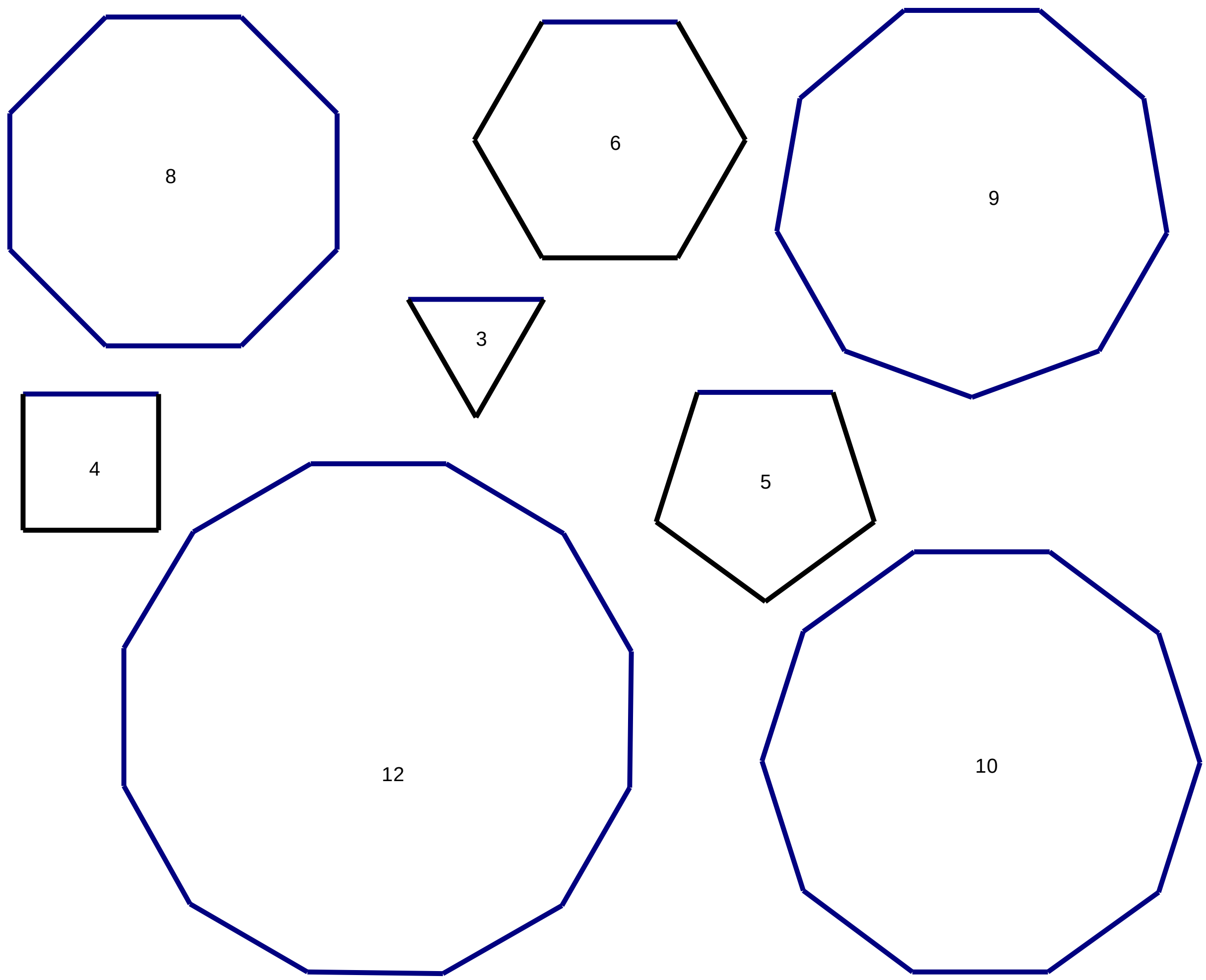
Other ideas to explore:

1. Will an arbitrary triangle tile the plane? If so, how could the edges be modified to produce an Escher-like tiling?
2. What about an arbitrary quadrilateral? Can you tile the plane with an arbitrary non-convex quadrilateral like this one?



3. What are some ways that a regular hexagonal tile could be modified to produce an Escher-like tessellation? Will an arbitrary hexagon tile the plane?

4. Go to the website http://www.mcescher.com/. Look at other Escher tilings and identify the process by which the tessellations were produced. Can you tell what the underlying tiling is? Templates for exploring regular and semi-regular tilings:



A good website for exploring semi-regular tilings is the NRICH Enriching Mathematics page, Semi-regular Tesselations (http://nrich.maths.org/4832).

Templates for exploring Escher-like tilings: