

The Power of Wishful Thinking And other problem solving strategies

Warm up Problem: The numbers 1 through 100 are written on the board. Each minute, you choose any two numbers x and y . Erase x and y and write $xy + x + y$ instead. After 99 minutes, there will be only one number left. Find all possible numbers that can occur at the end of this process.

What is wishful thinking? Besides being a fun pastime, it can be a useful tool in problem solving. The idea is simple: you have a problem that you don't know how to solve. What can you do? Well, you may be able to change the problem to one you can solve. If you are clever (or lucky) about choosing what you wish for, you may get an idea about how to solve the harder problem you started with. Here's a nice example: suppose you need to factor $x^4 + x^2 + 1$. It's a fourth degree polynomial, but notice that all the powers are even, so first thing we can do is re-write it as a quadratic in x^2 : $(x^2)^2 + x^2 + 1$. Now if we let $u = x^2$, then we have something much easier to think about: $u^2 + u + 1$. Trying to guess factors leads us nowhere, but it looks so close to a perfect square, doesn't it? If only the middle term was $2u$ instead of u . If only....., but wait! We notice that $u^2 + u + 1 = (u^2 + 2u + 1) - u = (u + 1)^2 - u$. Remember that u is x^2 ? This is the difference of squares! We can factor that! Now we finish the factorization: $(u + 1)^2 - u = (u + 1)^2 - x^2 = (u + 1 - x)(u + 1 + x)$. So finally, putting it all together we have: $x^4 + x^2 + 1 = (x^2 + 1 - x)(x^2 + 1 + x)$. A quick check of the discriminant tells us that both of these quadratics are irreducible and so we are done.

This is just one example of how wishful thinking can be used. There are many ways to change a given problem in a useful way to one that is easier to solve. Start wishing and see how many of the following problems you can solve.

1. Find the sum of the digits in the product $55,555,555,555 \times 999,999,999,999$
2. Show that 100^{100} can be written as the sum of 4 perfect cubes.
3. Show that a square can be inscribed in any triangle, *i.e.* show that there is a square with two vertices one side of the triangle and the other two vertices on the remaining two sides of the triangle.

