SET

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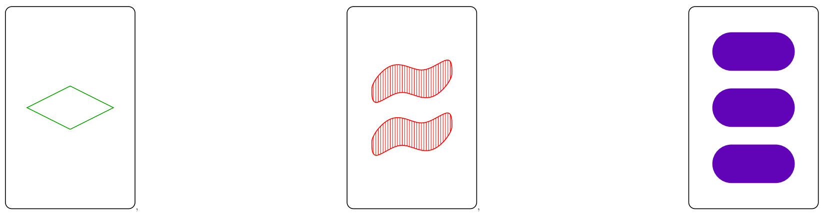
1. Introduction

Set (see http://www.setgame.com/) is a fun multi-player game played with a special deck of Set cards. Each card depicts one, two, or three shapes. All the shapes on a given card are the same and also share the same color and shading. The possible shapes are diamond, oval, and squiggle; the possible colors are red, green, and purple; and the possible shadings are plain, striped, and solid. Thus, each card has 4 characteristics: NUMBER, SHADING, COLOR, and SHAPE. For example, here are “one plain green diamond,” “two striped red squiggles,” and “three solid purple ovals.”



There is exactly one of each possible card in the Set deck, which leads to the question:

(1) How many cards in a Set deck?

Since there are three possible values for each of the four characteristics, there are a total of $3^4 = 81$ cards in a Set deck.

A Set is a collection of three cards such that for each characteristic, either all the cards in the trio share the same value for that characteristic, or else no two have the same value for it. For example, if the characteristic is SHAPE, then the three cards should all have the same shape or they should all have different shapes. Similarly for the other three characteristics. So, to determine if a trio is a Set one performs four checks, one for each of the four characteristics. If the trio passes all four of these tests then it is a Set.

(2) The three cards shown above form a Set. Why?

The game is played by dealing an array of 12 cards in the middle of the group of players. The first one to spot a Set in the array shouts “Set” and collects the three cards; then these cards are replaced and play continues. The person with the most Sets at the end is the winner.

Occasionally there will not be a Set amongst the 12 cards in the arrray. When all the players agree that there is no Set, an additional three cards are dealt. If a Set is found and removed from the 15 cards no additional cards are dealt. If no Set is found among the 15 cards, an additional three cards are dealt, and so on.

In practice, one rarely deals as many as 18 cards. We've never witnessed the necessity of dealing 21 cards (which is theoretically possible - see the last section of these notes).

It takes a little practice to identify Sets. So initial games may move slowly. However, with a group of 4 experienced players, the game moves at lightning speed. In order to compete in such a group, one has to learn to spot Sets without seemingly having to perform the four checks described above.

The game usually ends with 6 or 9 cards left in the middle which contain no Set. Very occasionally all the cards are used. We don't know the probability that a game of Set will end with all of the cards being used.

Novice players often misidentify Sets. The best way to overcome this seems to come from the following principle:

(3) Show that for every two cards there is a unique third card that will make a Set.

It is not hard to see that this must be true. Take the characteristic NUMBER. If the first two cards have the same number, then the third card must have that same number. If the first two cards have different numbers, then the third card must have a number different from the first two and there is only one possibility for that. Similarly for the other three characteristics.

For example, given the two cards "three striped red ovals" and "one plain green oval," the unique third card that will form a Set with these two is "two solid purple ovals."

(4) If you draw three cards at random, what is the probability that they form a Set?

Since any two cards uniquely determine a Set, if you draw two cards, then exactly one of the remaining 79 cards will join with the first two cards to make a Set. The probability of drawing it is 1/79.

(5) How many different Sets are there?

There are 81 choices for the first card, 80 choices for the second card, and one choice for the third card. Since the order of cards in a Set don't matter, we should divide by $3! = 6$. Thus there are $81 \cdot 80/6 = 1080$ different Sets.

(6) After playing a few rounds of Set, you'll notice that there are different kinds of Sets. What are they? Which kinds do you think are most and least common? Which seem to be the trickiest to spot?

There are actually four different kinds of Sets. They can be classified according to how many characteristics the cards share: zero, one, two, or three. The three-alike Sets are easiest to spot. An example of a three-alike Set is "one plain red squiggle," " two plain red squiggles," and "three plain red squiggles." The none-alike or all-different Sets are the most challenging to find. An example of an all-different Set is "one plain red oval," "two striped green squiggles," "three solid purple diamonds."

(7) Calculate the percentage of three-alike Sets, two-alike Sets, one-alike Sets, and all-different Sets.

If your calculation is correct you should have found that out of the 1080 total possible Sets the number of three-alike Sets is 108, or 10%, the number of two-alike Sets is 324, or

30%, the number of one-alike Sets is 432, or 40%, and the number of all-different Sets is 216 or 20%.

There are many more combinatorial or "counting" questions one can ask about Set (as well as many questions about the game's psychology!). For now, though, we want to turn to some other fascinating mathematical facets of the game.

2. Coordinatization

We can represent Set cards by 4-tuples of numbers, where each coordinate in the 4-tuple represents one of the characteristics. Let's arbitrarily put the coordinates in the order "number," "'shading," "color," and "shape." We'll assign values as follows:

Coordinate	Number	Shading	Color	Shape
1	1	plain	red	diamond
2	2	striped	green	oval
3	3	solid	purple	squiggle

So, for example, the card "2, striped, green, diamonds" becomes the point (2,2,2,1) in our coordinate system.

The cards on the previous page are "one, plain, green, diamond" which has coordinates $(1, 1, 2, 1)$; "two, striped, red, squiggles" which is $(2, 2, 1, 3)$; and "three, solid, purple, ovals" which is $(3, 3, 3, 2)$. Just for fun, let's add these up coordinate-by-coordinate:

$$(1, 1, 2, 1) + (2, 2, 1, 3) + (3, 3, 3, 2) = (6, 6, 6, 6).$$

(8) Try adding the coordinates of some other Sets and some non-Sets. What do you notice?

Let's describe this addition process a little more formally. To add or subtract in our system, we proceed coordinate by coordinate. For instance, (1,2,1,1)+(2,1,2,2,) = (1+2,2+1,1+2,1+2) = (3,3,3,3). Notice, however, that if the values of the coordinates in the 4-tuples are large enough, we end up with values that we can't interpret within our system, such as (2,1,3,1) + (2,3,3,1) = (2+2,1+3,3+3, 1+1) = (4,4,6,2). It would be nice to keep our system closed, so that if we add or subtract any number of 4-tuples we still only get coordinate values that we can interpret and link to cards if we want to.

An easy way to make this work is to interpret each coordinate value modulo (or "mod") 3. In mod 3 arithmetic, numbers that are multiples of 3 (e.g., -3, 0, 3, 6, ...) become 0s, numbers that differ from 1 by a multiple of 3 (e.g., -2, 1, 4, 7, ...) become 1s, and numbers that differ from 2 by a multiple of 3 (e.g., -1, 2, 5, 8, ...) become 2s. Here is an addition table mod 3:

+	-3	-2	-1	0	1	2	3
-3	0	1	2	0	1	2	0
-2	1	2	0	1	2	0	1
-1	2	0	1	2	0	1	2
0	0	1	2	0	1	2	0
1	1	2	0	1	2	0	1
2	2	0	1	2	0	1	2
3	0	1	2	0	1	2	0

Our coordinate system is closed under addition mod 3. Addition mod 3 is also commutative and associative.

(9) Now try adding up mod 3 the coordinates of the three cards in a Set. What do you get? Is the answer always the same? Why?

You should find that the sum of the cards in a Set is always (0,0,0,0). Take any characteristic of your Set. Since the three cards either all have the same value or all have different values of that characteristic, the sum of their coordinates for that characteristic is either 0+0+0, 1+1+1, 2+2+2, or 0+1+2. All of these add up to 0 mod 3. This holds true for all four characteristics, so the sum of the three cards is (0,0,0,0). Also, if three cards do not form a Set, then for at least one characteristic, the values of the coordinates are not all the same or all different, i.e. two of the cards have the same value for that characteristic and the other card has a different value. This means that the sum of the coordinates for that characteristic is 0+0+1, 0+0+2, 1+1+0, 1+1+2, 2+2+0, or 2+2+1. None of these sums to 0 mod 3.

In summary, we have found that three cards form a Set if and only if their sum mod 3 is (0,0,0,0). It can also be useful to rephrase this algebraically: Three cards A, B, and C form a Set if and only if A+B+C=0 mod 3.

3. The Geometry of Set

In the geometry of Set, each Set card is identified with a point in a 4-dimensional space. Think of each characteristic – number, shading, color, shape – as representing one dimension. This is a natural extension of our coordinate system.

One easy way to visualize this space is to sort your cards as follows: First, divide up the cards by color. You'll have a red pile, a purple pile, and a green pile, each of which will have 27 cards. Put the red pile on the left, the purple pile in the middle, and the green pile on the right. Now divide up each pile of 27 by shading, so that you have 9 plain cards, 9 striped cards, and 9 solid cards together from each color. Put the plain cards above the others, then the striped cards in the middle, and finally the solids. Now arrange each group of 9 cards into 3 rows and 3 columns so that their number characteristics increase going from left to right across each row, and so that the shape characteristics go from diamond to oval to squiggle down each column. You should end up with the picture of the 4-dimensional Set universe on the last page of this handout.

We have already said that each Set card is a point in this space. We have also shown that any two cards uniquely determine a third card that forms a Set. From these pieces of information, you can see that Sets correspond to lines in Set geometry, and that two cards (points) determine a Set (line).

Let's explore the geometry of Set a bit further:

(10) Pick out any three cards that don't form a Set. Place them so that they form a "corner." Now fill in all the Sets using a logical placement of the new cards. What's the most cards you can collect by repeating this process?

The answer you should come up with is 9 cards. This collection of cards, determined by the selection of 3 cards that do not form a Set – i.e., three non-collinear points – is a plane.

(11) How do we know that the procedure above works and that we don't end up with "odd cards out"?

The easiest way to see this is to look at it algebraically. Recall that the sum of the three cards in a Set is (0,0,0,0) modulo 3. We also phrased this as "Three cards A, B, and C form a Set if and only if A+B+C=0 mod 3." Say that you started with the cards **A**, **B**, and **D** and have used the procedure above to get:

A	**B**	C
D	E	F
G	H	I

Note that since A+B+C=0 and A+D+G=0, C=-A-B=2A+2B mod 3, and G=-A-D=2A+2D mod 3. Working from there, you should be able to write an equation for each of the other cards (E, F, H, and I) in terms of A, B, and D. This demonstrates that you only need three cards to determine this collection of 9 and that all 9 of the cards you end up with "belong" in this plane.

(12) How many Sets are there in a plane?

The easiest way to see this is to draw all the lines connecting the Sets in a plane, for example one of the 3 x 3 arrays displayed in the Set universe figure. Connect the cards in each row, each column, each diagonal, and then you've found all the Sets, right? Actually, if you look closely, you'll notice that there are a few Sets that form what we'll call "wrap-around" lines. In this geometry, to make a straight line, or Set, start at a point (card) and head in a direction. If the direction seems to take you out of the space, you wrap around and re-enter from the other side.

Counting the wrap-around lines as well as the rows, columns, and diagonals, you'll find that there are 12 Sets in a plane. You can see combinatorially that this is the maximum number of Sets you can have with 9 cards. If every pair of cards in the collection of 9 points to a third card that is also in the collection, then there are 9 ways to choose the first card, 8 ways to choose the second, and 1 to choose the third (since each third card in a Set is uniquely determined by the first two), so $9 \cdot 8 \cdot 1 = 72$, and then we have to divide by $3! = 6$ since the order of the cards doesn't matter, so 72/6=12.

(13) Take a plane and add one card. Complete all the Sets. How many cards do you come up with in your new collection? What does this collection correspond to geometrically?

You should end up with 27 cards, which is a 3-space or hyperplane in this geometry. Some easy 3-spaces to visualize are all the red cards, all the ones, all the ovals, all the solids, etc.

(14) We've seen that 2 cards (points) uniquely determine a line, 3 non-collinear points uniquely determine a plane, and 4 non-coplanar points uniquely determine a 3-space. How many cards are needed to determine the entire Set universe?

Only five cards are needed to determine the Set universe. (They can't all be in the same hyperplane.) Try it!

Let's use the term *closed* to describe a collection that has the property that whenever two points are in it, so is the third point which is on the line determined by those two. We can think of lines as 1-dimensional, planes as 2-dimensional, and 3-spaces (hyperplanes) as 3-dimensional; these are the closed subsets of the 4-dimensional Set universe. Individual points can be thought of as zero-dimensional.

Here are some questions investigating the relationships between these closed subsets:

(15) How many lines pass through a given point in the Set universe?
(16) How many planes pass through a given point?
(17) How many planes pass through a given line?
(18) How many different lines are there parallel to a given line?
(19) Show that given any 4 coplanar points, no three on a line, we can pair them up so that they determine two parallel lines, and we can also pair them up so that they form two intersecting lines.

One interesting consequence of being in a 4-dimensional universe is that there are skew planes!These are planes that are not parallel, yet they don't intersect.

(20) Give an example, using Set cards, of two planes that are parallel, two planes that intersect in a line, two planes that intersect in a point, and two planes that are skew.
(21) What are the possibilities for the intersection of a plane and a hyperplane? a line and a hyperplane? two hyperplanes?