Fun with Folding and Pouring

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Note: This lesson is based on Chapter 9, *Mathematics Galore!* by James Tanton, 2012, MAA, Washington DC.

Preliminary Folding Investigation

1. Take a strip of paper, fold it in half, and make a good crease at the midpoint position.

2. Open up the strip, and mark the crease “Midpoint”.

3. With the strip unfolded, fold the *left* end over to meet the “Midpoint”. Make a new crease halfway between the “Midpoint’ and the left end of the strip by folding the left end of the paper.

4. Open up the strip again, and mark the new crease “Fold 1”.

5. With the strip unfolded, make a new crease halfway between the “Fold 1” and the right end of the strip by folding the right end of the paper to meet the crease marked “Fold 1”.

6. Open up the strip again, and mark the new crease “Fold 2”.

7. Repeat, alternating left and right folds, with each fold made to the most recent crease mark. Mark each successive crease with $3, 4, \ldots$.

8. The sequence of crease marks seems to converge to two positions on the strip, what are they?

9. With a new strip of paper, repeat the experiment, except this time make the initial crease mark *anywhere* on the strip, not at the midpoint. The sequence of crease marks seems to converge to two positions on the strip, what are they this time?

Mathematical Analysis

Suppose the strip is one unit long, and the initial crease is at arbitrary position x ($0 < x < 1$ measured as a fraction of the length of the strip) as in the last step above.

1. Then a left fold to an arbitrary position x creates a new crease 1 at what position? (Express algebraically in terms of x.)

2. Now a right fold to a position x creates a new (even-numbered) crease at what position? (Express algebraically in terms of x.)

3. So with the second experiment, making the initial crease mark x *anywhere* on the strip, not at the midpoint, algebraically describe the position of the first 4 folds, left, right, left, right. (Hint: use compositions of functions.)

Aside on base 2

In base ten arithmetic, the decimal $0.abcd\ldots$ represents

$$\frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \frac{d}{10000} + \ldots.$$

Now think about base-2

1. In base two arithmetic, $0.abcd\ldots$ would represent what?

2. We can represent every real number x, $0 \leq x \leq 1$ in base two as $0.abcd\ldots$.

   (a) What is $\frac{3}{4}$ in base two?

   (b) What does $0.11111\ldots$ represent in base 2? (*Hint:* Remember geometric series!)

(c) What is $\frac{1}{3}$ in base two?

3. If
$$x = \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \dots$$
so $x = 0.abcd\dots$, then what is
$$2x$$

4. Describe what multiplying by 2 does to the "decimal point". (Maybe we should call it the "binary point" or the "binimal point"!)

5. In base two, if:
$$x = \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \dots$$
so $x = 0.abcd\dots$, then what is:
$$\frac{x}{2}$$

6. Describe what dividing by 2 does to the "decimal point". (Maybe we should call it the "binary point" or the

Back to Paper Folding

1. If the initial crease is at $x = 0.abcd\dots$ (in binary) then a left fold puts a new crease where?

2. A right fold to x puts a crease at
$$\frac{1}{2} + \frac{x}{2}$$
Describe the action of this in binary notation. (*Hint:* recall that $\frac{1}{2} = 0.1$.)

3. Thus if we make four right and left folds, where will be the latest crease?

4. If we make ten right and left folds, where will be the latest crease?

5. With more and more folds, what values will we be approaching?