1. The Problem

Can one remove one of the factorials from

\[ 1!2!3! \cdots 99!100! \]

so that what remains is a square?
2. Solution and Notes

The answer to this problem is 50!.

Here is why.

\[100! = 100 \times 99!, \quad 98! = 98 \times 97!, \quad 96! = 96 \times 95!\] and so on. So

\[1!2!3!4! \cdots 97!98!99!100! = 2(1!)^24(3!)^2 \cdots 98(97!)^2100(99!)^2\]

and thus the product is

\[2 \cdot 4 \cdots 98 \cdot 100 \times 1!2!^2 \cdots 97!^299!^2 = 2^{50} \times 50! \times 1!2!^2 \cdots 97!^299!^2.\]

Since \(2^{50} = 4^{25}\) if we remove 50! we have a square.

Some extensions.

If we think about 1!2!3! \cdots 99!100! we see it cannot be a square since 50! is not a square since 47 appears as a single factor.

So now let’s generalize and let’s give the product a name.

Let \(G(n) = 1!2!3! \cdots (n-1)!n\) and ask the question is \(G(n)\) ever a square if \(n \neq 1\)?

Note that \(G\) has the property that \(G(n) = n! \times G(n-1)\).

It doesn’t take too long to realize that if \(n\) is a multiple of 4, say \(4k\) then what we did before works and we can take out the \(2k!\) and we have a square. So this reduces the question to whether or not \(2^k!\) is a square. But it cannot be since there is always a prime factor between \(k\) and \(2k\) for \(k > 1\). This is Bertrand’s Postulate, conjectured in 1845 by Joseph Bertrand (1822-1900) and proved by Chebyshev (1821-1894) in 1850.

For other cases, let us go back to the example.

\(G(101) = G(100)101!\) is a square times 50! \(\times 101!\) and so cannot be a square. Here we even have that 101 is prime.

\(G(102) = G(100)101!102!\) is a square times 50! \(\times 101!102!\) and this last factor is a square times 50! \(\times 102\) and hence is not a square.

\(G(103) = G(100)101!102!103!\) is a square times 50! \(\times 102 \times 103!\) and thus it is not a square. Here 103! must have a prime between 51 and 102.
These arguments can be carried out in general. It takes a bit of work and makes use of Bertrand’s Postulate. A future question that follows from the very first posed question might be how many different ways can one of the factorials be taken out to get a square.