FACTORIALS AND SQUARES: NOTES AND EXTENSIONS

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1. The Problem

Can one remove one of the factorials from

$$1!2!3!\cdots 99!100!$$

so that what remains is a square?

2. Solution and Notes

The answer to this problem is 50!.

Here is why.

$100! = 100 \times 99!$, $98! = 98 \times 97!$, $96! = 96 \times 95!$ and so on. So

$$1!2!3!4! \cdots 97!98!99!100! = 2(1!)^2 4(3!)^2 \cdots 98(97!)^2 100(99!)^2$$

and thus the product is

$$2 \cdot 4 \cdot \cdots 98 \cdot 100 \times 1!^2 3!^2 \cdots 97!^2 99!^2 = 2^{50} \times 50! \times 1!^2 3!^2 \cdots 97!^2 99!^2.$$

Since $2^{50} = 4^{25}$ if we remove 50! we have a square.

Some extensions.

If we think about $1!2!3! \cdots 99!100!$ we see it cannot be a square since 50! is not a square since 47 appears as a single factor.

So now let's generalize and let's give the product a name.

Let $G(n) = 1!2!3! \cdots (n-1)!n!$ and ask the question is $G(n)$ ever a square if $n \neq 1$?

Note that G has the property that $G(n) = n! \times G(n-1)$.

It doesn't take too long to realize that if n is a multiple of 4, say $4k$ then what we did before works and we can take out the $2k!$ and we have a square. So this reduces the question to whether or not $2k!$ is a square. But it cannot be since there is always a prime factor between k and $2k$ for $k > 1$. This is Bertrand's Postulate, conjectured in 1845 by Joseph Bertrand (18221900) and proved by Chebyshev (18211894) in 1850.

For other cases, let us go back to the example.

$G(101) = G(100)101!$ is a square times $50! \times 101!$ and so cannot be a square. Here we even have that 101 is prime.

$G(102) = G(100)101!102!$ is a square times $50! \times 101!102!$ and this last factor is a square times $50! \times 102$ and hence is not a square.

$G(103) = G(100)101!102!103!$ is a square times $50! \times 102 \times 103!$ and thus it is not a square. Here 103! must have a prime between 51 and 102.

These arguments can be carried out in general. It takes a bit of work and makes use of Bertrand's Postulate. A future question that follows from the very first posed question might be how many different ways can one of the factorials be taken out to get a square.