

Compositions Teacher's Notes

The biggest goal here is to get the students to *organize* their work in such a way that everyone can tell when they've found all the compositions. The later work filling in the chart also suggests several ways to organize the work from the beginning.

Secondarily, they should stay on the lookout for ways to re-use previous work. For instance, if they are making 7 with first part 4, they should see that the ways to finish are exactly all the ways to make 3.

Selected hints, answers, and solutions

1. 1 is just 1.
2 is 2 or 1+1
3 is 3 or 2+1 or 1+2 or 1+1+1
4 is 4 or 3+1 or 2+2 or 2+1+1 or 1+3 or 1+2+1 or 1+1+2 or 1+1+1+1.
2. Maybe they'll see the pattern of doubling here. There are a lot of ways to see why! One is coming up in the chart organizing things by first part. Another nice way is to take all the compositions of the previous number, and either increase the last part by 1, or add a +1 to the end. This gives all the compositions of the next number, in two equal-sized groups: those with last part greater than 1, and those with last part equal to 1.
3. Each column is 1, 1, 2, 4, 8, 16, 32, ... because of that same reason: the next entry contains twice as many compositions because you can either increase the last part by 1 or add a new +1 part on the end. The exception to the doubling pattern at the beginning is because when the composition has only one part, you can't increase it by 1 without changing which column you'll be counted in.

Another way to understand this pattern is that if you're making, for instance, 7 with first part 3, all the ways to finish are exactly the compositions of 4, so each entry is the sum of a previous row of this chart.

4. This is the Fibonacci sequence! One way to see why is that to make, for instance, 7 with only odd parts, you can either take a composition of 6 with odd parts and put a +1 on the end, or take a composition of 5 with odd parts and increase the last part by 2. This shows that the number of ways for 7 is the same as the number of ways for 6 plus the number of ways for 5, as in the Fibonacci sequence.

(For some students who seem ready for it, you should encourage them to explain why every valid composition of 7 is counted exactly once by this method. How do we know there aren't any left out? This same question applies throughout, too)

5. This is Pascal's Triangle. One way to see it is that the number of ways of, for instance, making 7 with 4 parts is to start with 6 in 3 parts (and put a +1 on the end for a fourth part) or to start with 6 in 4 parts and increase the last part by 1. This shows that you form each entry by adding two entries in the previous row, just as in Pascal's triangle.

You can also argue using the tally marks: If you're making 7 with 4 parts, then you need to write your tally marks $| \quad | \quad | \quad | \quad | \quad | \quad |$ and choose 3 of the 6 gaps to insert a plus sign into.

6. This one seems to be a lot trickier to get the hang of. There's an almost-Fibonacci pattern going on here (which might be easier to see if we listed the number of compositions whose largest part is less than or equal to a number, instead of exactly equalling it).

One pattern to recognize is that after a first few “unusual” terms, the diagonals eventually become constant. For instance, the diagonal of compositions of n with largest part $n-2$ eventually becomes a constant 5. That's because, for example, if we're making 8 with largest part 6, we have $8+2$, $2+8$, $8+1+1$, $1+8+1$, and $1+1+8$.

But what is that constant? What is the pattern in 1, 2, 5, 12, ...? It turns out that the pattern is to double and add successive powers of 2. Double 1 to get 2, double and add 1 to get 5, double and add 2 to get 12, double and add 4 to get 28, double and add 8 to get 64, and so on. Can you see why this pattern works? How far down the diagonal do you need to get for the “eventually constant” to happen?

There's also some other counting to be done, of course: how do you explain the specific values before the diagonals become constant?

All the compositions of 7 with largest part 3, for example, must end with a 1, 2, or 3. So, you can form them by taking a composition of 6 with largest part 3 and putting a +1 at the end; or a composition of 5 with largest part 3 and putting a +2 at the end; or a composition of 4 with largest part 1, 2, or 3 and then adding a +3 to the end. Thus, this new number is the sum of five previous numbers. This shows why the “largest part 2” column is (almost) Fibonacci.

Extensions: You can put other rules on your compositions, like we did with the “only odd parts” – what if all the parts must be unequal? What if the parts must be consecutive integers? What if the parts are all powers of 2?

If you're very brave, you can open the whole topic of partitions, where the order doesn't matter (so $1+2$ and $2+1$ are the same partition of 3, which now has only 3 partitions instead of 4 compositions). That could take years to explore fully.