Games! June 2012

HtRaMTC
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Seven Mathematical Games

For all but #7 below, two players alternate turns. The winner is the last player who makes a legal move. See if you can find a winning strategy for one of the players. Try to prove that your strategy works. And, always, try to generalize!

1 *Breaking the Bar*. Start with a rectangular chocolate bar which is 6×8 squares in size. A legal move is breaking a piece of chocolate along a single straight line bounded by the squares. For example, you can turn the original bar into a 6×2 piece and a 6×6 piece, and this latter piece can be turned into a 1×6 piece and a 5×6 piece. What about the general case (the starting bar is $m \times n$)?

2 *Takeaway*. A set of 16 pennies is placed on a table. Two players take turns removing pennies. At each turn, a player must remove between 1 and 4 pennies (inclusive).

3 *Putdown*. Each player takes turns placing a penny on the surface of a rectangular table. No penny can touch a penny that is already on the table. The table starts out completely bare.

4 *Puppies and Kittens, aka Wythoff's Nim*. We start with a pile of 7 kittens and 10 puppies. Two players take turns; a legal move is removing any number of puppies or any number of kittens or an equal number of both puppies and kittens.

5 *Color the Grids*. You start with an $n \times m$ grid of graph paper. Players take turns coloring red one previously uncolored unit edge of the grid (including the boundary). A move is legal as long as no closed path has been created.

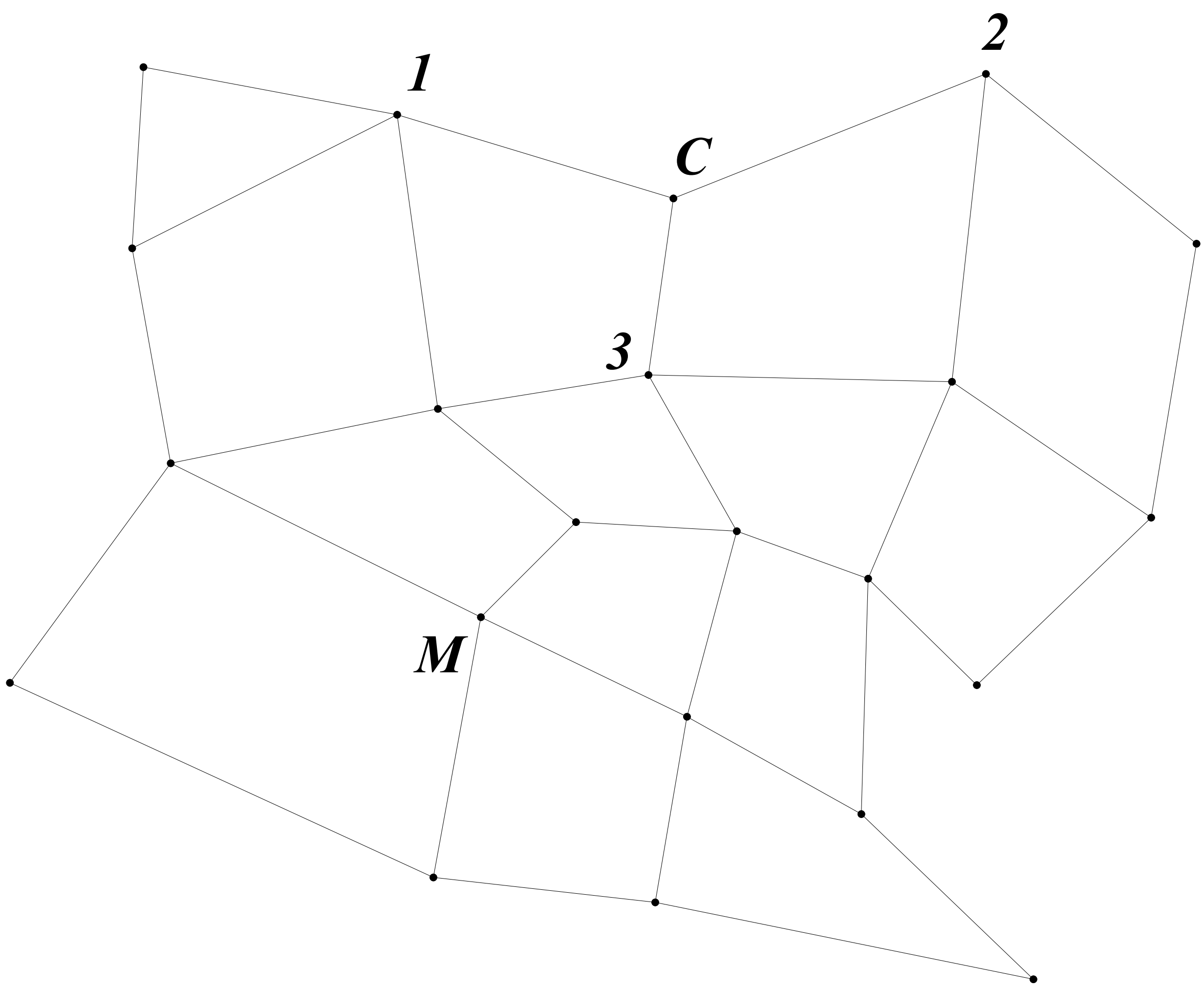
6 *Nim*. Start with several piles of beans. A legal move consists of removing one or more beans from a pile.

(a) Verify that this game is *very* easy to play if you start with just one pile, for example, of 17 beans.

(b) Likewise, if the game starts with two piles, the game is quite easy to analyze. Do it!

(c) But what if we start with three or more piles? For example, how do we play the game if it starts with three piles of 17, 11, and 8 beans, respectively? What about four piles? More?

7 *Cat and Mouse*. (Adapted from Ravi Vakil's excellent book, *A Mathematical Mosaic*.) A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below.



Initially, the cat is at the point labeled C; the mouse is at M. The cat goes first, and can move to any neighboring point connected to it by a single edge. Thus the cat can go to points 1, 2, or 3, but no others, on its first turn. The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win?

Advanced Digression: Puppies and Kittens and the Golden Ratio

The problems below will help you to discover the stunning fact that the golden ratio is intimately involved with the Puppies and Kittens game. Let (x_n, y_n) be the Puppies and Kittens oasis satisfying $y_n - x_n = n$. For example,

$$x_1 = 1, y_1 = 2, \quad x_2 = 3, y_2 = 5, \quad x_3 = 4, y_3 = 7.$$

Our goal is to show, for all $n = 1, 2, 3, \ldots$, that $x_n = \lfloor n\tau \rfloor$, and thus $y_n = \lfloor n(\tau + 1) \rfloor$, where τ is the famous, ubiquitous *Golden Ratio*:

$$\tau = \frac{1 + \sqrt{5}}{2}.$$

1 Show that $\tau^2 = \tau + 1$ and, thus,

$$\frac{1}{\tau} + \frac{1}{\tau + 1} = 1.$$

2 Two disjoint sets whose union is the natural numbers $\mathbf{N} = \{1, 2, 3, \ldots\}$ are said to *partition* $\mathbf{N}$. In other words, if A and B partition $\mathbf{N}$, then every natural number is a member of exactly one of the sets A, B. No overlaps, and no omissions.

Let (x_n, y_n) be the Puppies and Kittens oasis satisfying $y_n - x_n = n$. For example,

$$x_1 = 1, y_1 = 2, x_2 = 3, y_2 = 5.$$

Verify that the two sets

$$A = \{x_1, x_2, x_3, \ldots\} \text{ and } B = \{y_1, y_2, y_3, \ldots\}$$

partition the natural numbers.

3 Let α be a positive real number. Define the set of *multiples* of α to be the positive integers

$$\{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots\}.$$

For example, if $\alpha = 2$, then the multiples of α are just the even positive integers. Notice that α need not be an integer, or even rational.

Does there exist α such that the multiples of α are the *odd* positive integers?

4 Suppose that there are two numbers α, β whose sets of multiples partition the natural numbers. In other words, every natural number is equal to the floor of an integer times *exactly one* of α or β, and there are no overlaps.

(a) Prove that both α and β are greater than 1.

(b) Suppose that $1 < \alpha < 1.1$. Show that $\beta \geq 10$.

(c) Prove that

$$\frac{1}{\alpha}+\frac{1}{\beta}=1.$$

Hint: How many numbers less than or equal to 2010 are multiples of 7? How many are multiples of 11?

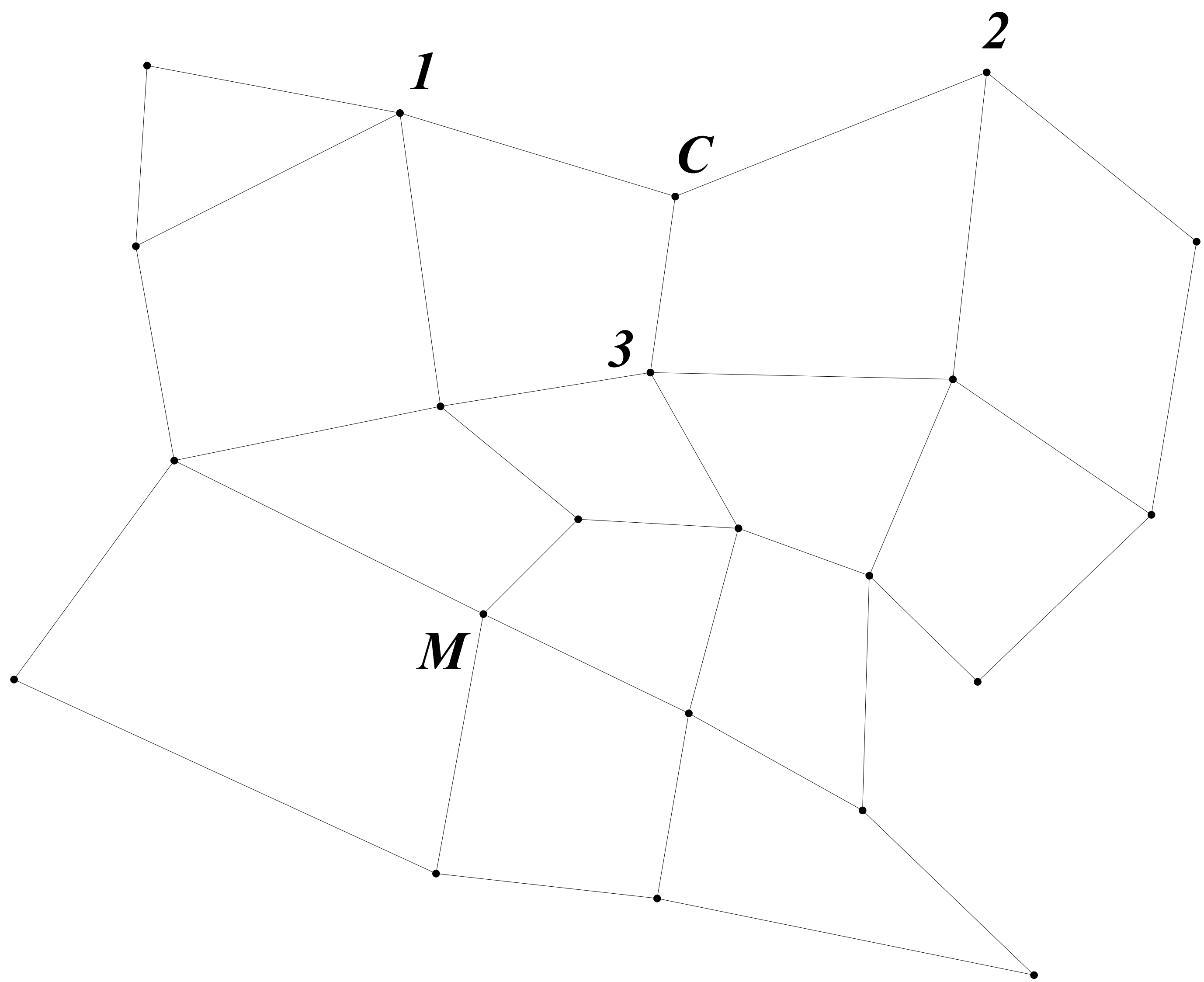
(d) Prove that α must be irrational (and hence by (a), β must also be irrational).

5 Prove the converse of the above problem; i.e., if α,β are positive irrational numbers satisfying

$$\frac{1}{\alpha}+\frac{1}{\beta}=1,$$

then the multiples of α and the multiples of β partition the natural numbers.

6 Why does this now show that $x_n=\lfloor n\tau\rfloor$, and thus $y_n=\lfloor n(\tau+1)\rfloor$?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