Transfinite Numbers
NY Math Circle Presentation
by Martin Rudolph

Is infinity the largest number? Consider an argument between two kids in the 1950s:
1: I dare you.
2: I double dare you.
1: I triple dare you.
2: I dare you times a million (said with a smirk)
1: I dare you times infinity (with finality)
2: I dare you times infinity *plus one!!!* (uh oh)
(Of course, in the film *A Christmas Story*, they all lose to a “double dog dare.”)

DEFINITIONS & NOTATION

1. A **set** is any collection of distinguishable objects.
2. Set *A* is a **subset** of set *B*, $A \subseteq B$, *iff* $x \in A \Rightarrow x \in B$ (every element of *A* is an element of *B*).
3. The **cardinal number of finite set *A*, #*A*,** is the number of elements in set *A*.
e.g. If *A* = {-3, 0, 2, 7}, then #*A* = 4.
4. A **1-1 correspondence between two sets** is a pairing of their elements such that *each* element of each set is paired with *exactly* one element of the other set.
e.g. If *A* = {-3, 0, 2, 7} and *B* = {a, b, c, d}, then an example of a 1-1 correspondence is -3↔a, 0↔d, 2↔b, and 7↔c. How many possible 1-1 correspondences are there?
5. Two sets, *A* and *B*, are **equivalent**, $A \sim B$, *iff* there exists a 1-1 correspondence between them. In other words, they have an equal number of elements (same size). In the above example, $A \sim B$. Notes: Equivalence requires the existence of *at least one* 1-1 correspondence. If we added one element to set *B*, then such a 1-1 correspondence would not exist and they would not be equivalent.
6. ***N*** = set of natural numbers; ***Z*** = set of integers; ***W*** = set of non-negative integers; ***E*** = set of even natural numbers; ***Q*** = set of rationals; $\boldsymbol{Q}^+$ = set of positive rationals; ***R*** = set of reals; $\boldsymbol{R}^{(0,1)} = \{x \in R \mid 0 < x < 1\}$, open unit interval
7. #***N*** = $\aleph_0$ (“aleph-null”)
8. An infinite set is **countable** or **(d)enumerable** *iff* it is equivalent to ***N***. Note: It can be shown that a countable set is “listable” (*i.e.*, it is possible to list all elements of the set such that each element appears in the list *exactly once*).
9. Equivalence of sets is a transitive relation. (Also has identity and symmetric properties; therefore, applying the Principal of Maximum Confusion, equivalence of sets is an equivalence relation.)

QUESTIONS FOR TODAY

(Today's discussion is dedicated to the great Dane, Georg Cantor, who was pretty great in math, but a dog in spelling...he always misspelled "George.")

1. A small discussion on small large numbers. (Is it possible for a single human mind to be "larger" than the observable universe? I say "yes, in a way.")

2. Are all infinite sets equivalent?

$$W \sim N? \quad E \sim N? \quad Z \sim N? \quad Q^{+} \sim N? \quad R^{(0,1)} \sim N?$$

In each case, why would we think "yes?"   Why would we think "no?"

3. $\overline{S} =$ the set of all subsets of set *S*, the "power set of set *S*." So what?

4. What is the Continuum Hypothesis?

QUESTIONS FOR TOMORROW

1. Prove: For any finite set *A* which has *n* ($n \in N$) elements, has $\#\overline{A} = 2^{n}$. Try Math Induction. (Then again, using math induction, one can "prove" that all humans are male! Can we trust math induction?) What if set *A* is infinite?

2. Prove: The set of reals is equivalent to the set of positive reals, $R \sim R^{+}$. (Curiously, it is easier to find the required 1-1 correspondence, of which there are *many*, if one imposes certain qualities on the 1-1 correspondence, thereby *reducing* the number of those that are acceptable!)

3. Prove: The open unit interval is equivalent to the open unit square.

$$R^{(0,1)} \sim \{(x,y) \mid x \in R, y \in R, 0 < x < 1, 0 < y < 1\}$$

READING SAMPLER

1. Birkhoff, Garrett and Saunders MacLane, *A Survey of Modern Algebra*, The Macmillan Co., 1953, 356–369. Theorems and proofs.

2. Courant, Richard and Herbert Robbins, *What is Mathematics?*, Oxford University Press, 1941, 77–88. A classic.

3. Kline, Morris: *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972, 992–1004. Contains some historical anecdotes.