The Biggest, the Smallest…

Tatiana Shubin
shubin @math.sjsu.edu

It is often interesting to see what is the biggest or the smallest value of something. Let's take a look – and then a better look – at some such questions.

Just to warm up:

- What is the biggest number? The smallest one?
- What is the biggest value of the sum of a positive real number and its reciprocal? What is the smallest one?
- Of a 100 balls contained in a box there are 28 red, 20 green, 12 yellow, 20 blue, 10 white, and 10 black ones. What is the smallest number of balls that we have to take out of the box to guarantee that there are 15 balls of the same color among them? (Of course, you can think instead of a 15-legged other-planet intelligent spider who likes wear socks of the same color and keeps his one hundred multicolor socks all in a heap in his drawer…)

The rest of the problems tonight will be geometric.

1. (a) How large can an angle of a triangle be?
   (b) How large can the largest angle of a triangle be?
   (c) How large can the smallest angle of a triangle be?
   (d) Same questions as above, only now we're interested in knowing how small instead of how large.

2. On a plane, there is a line and two points *A* and *B* at the same side off the line. Find the point *M* on the line with the property that the sum of the distances $AM + BM$ is the smallest.

Note: What if we ask the same question about three points off the line instead of two?

3. There is a river between two towns; each town is at some distance from the river, and the river banks are two parallel lines. Where a bridge across the river should be built in order to minimize the length of the road connecting the towns?

4. Same question as above – only now there are two parallel rivers of different widths between the two towns, and, of course, two bridges have to be built.

5. Given a point M inside an acute angle, find two points on the sides of the angle, *A* and *B* (exactly one on each side) so that the perimeter of the triangle *MAB* is as small as possible. (In 'practical' terms, suppose your house is at the point M, one side of the angle is a river, and another side, a road. Where should you plant a tree at the road and where should you get to the river so that your total trip – from your home to the river to get

bucketful of water, then to the tree to water it, and finally back home again – is as short as possible?)

6. Given two points, M and N, inside an acute angle, find two points on the sides of the angle, A and B (exactly one on each side) so that the perimeter of the quadrilateral with vertices at the points M, A, B, N is as small as possible.

7. For a convex quadrilateral $ABCD$, find a point inside it with the property that the sum of the distances from this point to all vertices of the quadrilateral is the smallest.

Note: What if we want to find such a point inside a triangle?

8. Triangle ABC has $AB = AC = 5$ cm and $BC = 6$ cm. From any point P, inside or on the boundary of this triangle, line segments are drawn at right angles to the sides; the lengths of these line segments are x, y and z cm.

(a) Find the largest possible value of the total $x + y + z$ and find the positions of P where this largest total occurs.
(b) Find the smallest value of the total $x + y + z$ and find the positions of P where this smallest total occurs.
(c) What if ABC is a general (non-isosceles) triangle?

9. *Fagnano's problem*: Given an acute-angled triangle ABC, find the triangle with the smallest perimeter whose vertices lie on the edges of the triangle ABC.

Note: a nice solution to this problem – and many other interesting problems – can be found in *Four Points on a Circle*, by Tom Davis, http://www.geometer.org/mathcircles/fourpoints.pdf

10. [*Calculus*, by James Stewart, p.332] A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer. How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle subtended at his eye by the painting?)

Note: Of course, we don't need calculus to solve this problem! Instead, we should recall a couple of beautiful facts about angles with the vertex on a circle, and with a vertex outside a circle. Anyone, ready to show a solution?

Are you tired of the angles and distances? Here are problems with a different flavor.

11. Place a number of points on the plane in such a way that no three of them are collinear, and no four of them are vertices of a convex quadrilateral. What is the largest number of points for which such a placement is possible?

12. Place a number of points on the plane in such a way that no three of them are collinear, and in every triangle formed by these points each angle is less than 120 degrees. What is the largest number of points for which such a placement is possible?