Let’s imagine that we introduce a new coin system. Instead of using pennies, nickels, dimes, and quarters, let’s say we agree on using only 4-cent and 7-cent coins. One might point out the following flaw of this new system: certain amounts cannot be exchanged, for example, 1, 2, or 5 cents. On the other hand, this deficiency makes our new coin system more interesting than the old one, because we can ask the question: “which amounts can be exchanged?” We will see shortly that there are only finitely many integer amounts that cannot be exchanged using our new coin system. A natural question, first tackled by Ferdinand Georg Frobenius and James Joseph Sylvester in the 19' th century, is: “what is the largest amount that cannot be exchanged?” As mathematicians, we like to keep questions as general as possible, and so we ask: given coins of denominations $a$ and $b$—positive integers without a common factor—can you give a formula $g(a, b)$ for the largest amount that cannot be exchanged using the coins $a$ and $b$? This problem and its generalization for coins $a_1, a_2, \ldots, a_n$ is known as the Frobenius coin-exchange problem.

To study the Frobenius number $g(a, b)$, we use the Euclidean Algorithm. For integers $a$ and $b$ that have no common factor, this algorithm yields integers $x$ and $y$ such that $ax + by = 1$.

1. Find $g(4, 7)$.
2. Show that $g(5, 11) = 39$.
3. Find $x$ and $y$ such that $4x + 7y = 1$.
4. Find another $x$ and $y$ such that $4x + 7y = 1$.
5. Find $x$ and $y$ such that $5x + 11y = 1$.
6. Find $x$ and $y$ such that $5x + 11y = 39$.
7. Show that, if $t$ is a given integer, we can always find integers $x$ and $y$ such that $4x + 7y = t$. Generalize to two coins $a$ and $b$ with no common factor.
8. Show that, if $t$ is a given integer, we can always find integers $x$ and $y$ such that $4x + 7y = t$ and $0 \leq x \leq 6$. Generalize to two coins $a$ and $b$ with no common factor.
9. Show that the following recipe for determining whether or not a given amount $t$ can be changed (using the coins 4 and 7) works: Given $t$, find integers $x$ and $y$ such that $4x + 7y = t$ and $0 \leq x \leq 6$. Then $t$ can be changed precisely if $y \geq 0$. Generalize to two coins $a$ and $b$ with no common factor.
10. Use the previous argument to re-compute $g(4, 7)$. Generalize your argument to compute $g(a, b)$, for any two coins $a$ and $b$ with no common factor.
11. Prove that exactly half of the amounts between 1 and $(a - 1)(b - 1)$ can be changed.
Now it’s time for something new. Every infinite sequence \((a_0, a_1, a_2, \ldots)\) comes with a handy analytic gadget, namely its generating function, which is defined as

\[ g(x) = \sum_{k=0}^{\infty} a_k x^k. \]

If you know some Analysis (and you don’t have to know any Analysis for these exercises), this looks like a power series, however, we don’t have to worry about convergence of this series, but rather treat it as a formal power series. In the course of the exercises, you will get a feeling for what this means.

(1) Show that \(1 + x + x^2 + x^3 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}\) for any number \(x\). Conclude that the infinite sum \(1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}\) (if we worry about convergence, we should demand that \(|x| < 1\)).

We just computed the generating function for the sequence \(a_k\) consisting of all 1’s:

\[ \sum_{k \geq 0} x^k = \frac{1}{1-x}. \]

Compute the sequence \((a_k)\) that gives rise to the generating function \(\sum_{k \geq 0} a_k x^k = \left(\frac{1}{1-x}\right)^2\), by looking at the product \((1 + x + x^2 + x^3 + \cdots) \cdot (1 + x + x^2 + x^3 + \cdots)\). If you look at the result, can you think of a different way to compute \((a_k)\)?

(2) Now we define a sequence recursively. Namely, we set \(a_0 = 0\) and \(a_{n+1} = 2a_n + 1\) for \(n \geq 0\).

(a) Conjecture a formula for \(a_k\) by experimenting.

(b) Now put the sequence \((a_k)\) into a generating function \(g(x)\) and find a formula for \(g(x)\) by utilizing the recursive definition of \(a_k\).

(c) Expand your formula for \(g(x)\) into partial fractions, and use the result to prove your conjectured formula for \(a_k\).

(3) We define a second recursive sequence by setting \(a_0 = 1\) and \(a_{n+1} = 2a_n + n\) for \(n \geq 0\). Find a formula for \(a_k\).

It’s time to go back to the Frobenius problem. Let us introduce the counting sequence

\[ r_k = \# \left\{ (m, n) \in \mathbb{Z}^2 : m, n \geq 0, \ ma + nb = k \right\}. \]

In words, \(r_k\) counts the representations of \(k \in \mathbb{Z}_{\geq 0}\) as nonnegative linear combinations of \(a\) and \(b\). Thus, \(r_{ab-a-b} = 0\), and \(ab - a - b\) is the largest integer \(k\) for which \(r_k = 0\).

(1) Prove that \(r_{k+ab} = r_k + 1\).

(2) Compute the generating function for the sequence

\[ a_k = \begin{cases} 1 & \text{if } k \text{ is a multiple of } 7, \\ 0 & \text{otherwise.} \end{cases} \]
(3) Prove that, for \( r_k = \# \{(m, n) \in \mathbb{Z}^2 : m, n \geq 0, ma + nb = k \}; \)

\[
\sum_{k \geq 0} r_k x^k = \left( \frac{1}{1 - x^a} \right) \left( \frac{1}{1 - x^b} \right).
\]

(4) Now let \( s_k = \begin{cases} 1 & \text{if } k \text{ can be changed,} \\ 0 & \text{otherwise.} \end{cases} \) Prove that

\[
\sum_{k \geq 0} s_k x^k = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)}.\]

A few remarks

The simple-looking formula for \( g(a, b) \) that you have found inspired a great deal of research into formulas for the general Frobenius number \( g(a_1, a_2, \ldots, a_d) \), with limited success: While it is safe to assume that the formula for \( g(a, b) \) has been known for more than a century, no analogous formula exists for \( d \geq 3 \). The case \( d = 3 \) is solved algorithmically, i.e., there are efficient algorithms to compute \( g(a, b, c) \), and in form of a semi-explicit formula. The Frobenius problem for fixed \( d \geq 4 \) has been proved to be computationally feasible, but no efficient practical algorithm for \( d = 4 \) is known.

A second classic theorem for the case \( d = 2 \), which you have proved and Sylvester published as a math problem in the Educational Times more than a century ago [2], says that exactly half of the amounts between 1 and \((a - 1)(b - 1)\) cannot be changed using the coins \( a \) and \( b \).

For more, we refer to a research monograph on the Frobenius problem [1]; it includes more than 400 references to articles written about the Frobenius problem.

References
