HYPERBINARY NUMBERS PROOFS KEY

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§1. HYPERBINARY NUMBERS

Claim 1. $b(n) = b(2n+1)$

Proof. Given a hyperbinary representation of n we can obtain a hyperbinary representation of $2n+1$ by adding a final 1 to the right end. Conversely, any hyperbinary representation of $2n+1$ necessarily has a 1 at its right end. Deleting this final 1 gives a hyperbinary representation of n. □

Claim 2. $b(2n) = b(n) + b(n-1)$

Proof. Suppose we start with a representation of $2n$. Since $2n$ is even, the hyperbinary representation must end in a 0 or a 2. If it ends in a 0, then chopping off this last 0 results in a representation of n. If it ends in a 2, then chopping off this 2 has the effect of subtracting 2 then dividing by 2. Hence it yields a representation of $n-1$. □

§2. CALKIN–WILF TREE

Claim 3. *If a node is labelled $\frac{r}{s}$ and the node is a left child, its parent labelled $\frac{r}{s-r}$. If a node is labelled $\frac{r}{s}$ and it is right child, its parent labelled $\frac{r-s}{s}$.*

Proof. The two expressions are simply the inverses of the left-child and right-child expressions given in the Calkin–Wilf tree rules. □

Claim 4. *Every number in the tree is a reduced fraction*

Proof. Suppose some node N is labelled $\frac{r}{s}$ and that r and s share a common factor. Let a be this common factor, so $a > 1$, and a divides both r and s. Furthermore, assume that this node N is at the highest possible level (least possible level index). Then the parent N' of N is either labelled $\frac{r-s}{s}$ or else $\frac{r}{s-r}$. But now a divides $r-s$ and a divides $s-r$, so N' is labelled with a fraction that is not in lowest terms. This contradicts that N was chosen to be at the highest possible level. □

Claim 5. *Every positive rational number appears in the Calkin–Wilf tree.*

Proof. Suppose the fraction $\frac{r}{s}$ is a fraction in lowest terms that does not appear in the tree. Furthermore, pick this fraction with the smallest possible sum $r+s$. Either $r < s$ or $s < r$. Then $\frac{r}{s}$ is either the child of $\frac{r}{s-r}$ or $\frac{r-s}{s}$. But by our assumption, since each of these has a sum smaller than $r+s$, each of these appears in the tree. Thus $\frac{r}{s}$ does too. □

Claim 6. *No number appears more than once in the Calkin–Wilf tree.*

Proof. Suppose the fraction $\frac{r}{s}$ is a fraction in lowest terms that appears in the tree more than once. Furthermore, pick this fraction with the smallest possible sum $r + s$. Again $\frac{r}{s}$ is either the child of $\frac{r}{s-r}$ or $\frac{r-s}{s}$. But it is not the child of both, as the choice of parent is determined by whether $r < s$ or $s < r$. So each of these instances of $\frac{r}{s}$ has the same parent. But then the parent of $\frac{r}{s}$ is an example of a fraction that appears in the tree more than once, and it has a sum smaller than $r + s$, a contradiction. □

§3. BRINGING THE TWO TOGETHER

Claim 7. *The denominator of node n is the numerator of node $n + 1$.*

Proof. Case I: We are looking at the left and right children of the same parent. In this case, if the parent is $\frac{r}{s}$, then the denominator if the left child and the numerator of the right child are both defined to be $r + s$.

Case II: We are looking at the end of one row followed by the beginning of the next row. In this case, the denominator of the end of a row and the numerator of the begining of a row are both 1.

Case III: We are looking at the right child of parent $\frac{a}{b}$ followed by the left child of parent $\frac{c}{d}$. For example, look at the case in the tree of $\frac{7}{3}$ followed by $\frac{3}{8}$, whose parents are $\frac{a}{b} = \frac{4}{3}$ and $\frac{c}{d} = \frac{3}{5}$. But this example shows how the proof will work, too. We can inductively assume that the claim is true for fractions at the level of $\frac{a}{b}$ and $\frac{c}{d}$. In other words $b = c$. Thus the right child of $\frac{a}{b}$ has denominator b, and the left child of $\frac{c}{d}$ has numerator $c = b$. This the theorem is true in this case, as well. □

Claim 8. *The fraction label of node n has the form $f(n)/f(n+1)$ for some sequence $f(n)$.*

Proof. This follows immediately from the previous claim. □

Claim 9. *The sequence $f(n)$ is exactly the same as the sequence $b(n)$ explored earlier.*

Proof. What is the left child of node n? The answer is always $2n + 1$, and this is a fun exercise on its own if there is time. This means the left child of the fraction labelled $\frac{f(n)}{f(n+1)}$ is the fraction labelled $\frac{f(2n+1)}{f(2n+2)}$. But then the definition of 'left child' tells us that $f(2n + 1) = f(n)$ and $f(2n) = f(n) + f(n + 1)$. Do these statements look familiar? They should, because these are the same statement that defined our hyperbinary sequence $b(n)$! □

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