HYPERBINARY NUMBERS PROOFS KEY

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§1. HYPERBINARY NUMBERS

Claim 1. $b(n) = b(2n + 1)$

Proof. Given a hyperbinary representation of $n$ we can obtain a hyperbinary representation of $2n + 1$ by adding a final 1 to the right end. Conversely, any hyperbinary representation of $2n + 1$ necessarily has a 1 at its right end. Deleting this final 1 gives a hyperbinary representation of $n$. □

Claim 2. $b(2n) = b(n) + b(n - 1)$

Proof. Suppose we start with a representation of $2n$. Since $2n$ is even, the hyperbinary representation must end in a 0 or a 2. If it ends in a 0, then chopping off this last 0 results in a representation of $n$. If it ends in a 2, then chopping off this 2 has the effect of subtracting 2 then dividing by 2. Hence it yields a representation of $n - 1$. □

§2. CALKIN–WILF TREE

Claim 3. If a node is labelled $\frac{r}{s}$ and the node is a left child, its parent labelled $\frac{r}{s-r}$ If a node is labelled $\frac{r}{s}$ and it is right child, its parent labelled $\frac{r-s}{s}$.

Proof. The two expressions are simply the inverses of the left-child and right-child expressions given in the Calkin–Wilf tree rules. □

Claim 4. Every number in the tree is a reduced fraction

Proof. Suppose some node $N$ is labelled $\frac{r}{s}$ and that $r$ and $s$ share a common factor. Let $a$ be this common factor, so $a > 1$, and $a$ divides both $r$ and $s$. Furthermore, assume that this node $N$ is at the highest possible level (least possible level index). Then the parent $N'$ of $N$ is either labelled $\frac{r-s}{s}$ or else $\frac{r}{s-r}$. But now $a$ divides $r - s$ and $a$ divides $s - r$, so $N'$ is labelled with a fraction that is not in lowest terms. This contradicts that $N$ was chosen to be at the highest possible level. □

Claim 5. Every positive rational number appears in the Calkin–Wilf tree.

Proof. Suppose the fraction $\frac{r}{s}$ is a fraction in lowest terms that does not appear in the tree. Furthermore, pick this fraction with the smallest possible sum $r + s$. Either $r < s$ or $s < r$. Then $\frac{r}{s}$ is either the child of $\frac{r}{s-r}$ or $\frac{r-s}{s}$. But by our assumption, since each of these has a sum smaller than $r + s$, each of these appears in the tree. Thus $\frac{r}{s}$ does too. □
Claim 6. No number appears more than once in the Calkin–Wilf tree.

Proof. Suppose the fraction \( \frac{r}{s} \) is a fraction in lowest terms that appears in the tree more than once. Furthermore, pick this fraction with the smallest possible sum \( r + s \). Again \( \frac{r}{s} \) is either the child of \( \frac{r-s}{s} \) or \( \frac{r}{r-s} \). But it is not the child of both, as the choice of parent is determined by whether \( r < s \) or \( s < r \). So each of these instances of \( \frac{r}{s} \) has the same parent. But then the parent of \( \frac{r}{s} \) is an example of a fraction that appears in the tree more than once, and it has a sum smaller than \( r + s \), a contradiction. □

§3. BRINGING THE TWO TOGETHER

Claim 7. The denominator of node \( n \) is the numerator of node \( n + 1 \).

Proof. Case I: We are looking at the left and right children of the same parent. In this case, if the parent is \( \frac{a}{b} \), then the denominator if the left child and the numerator of the right child are both defined to be \( r + s \).

Case II: We are looking at the end of one row followed by the beginning of the next row. In this case, the denominator of the end of a row and the numerator of the beginning of a row are both 1.

Case III: We are looking at the right child of parent \( \frac{a}{b} \) followed by the left child of parent \( \frac{c}{d} \). For example, look at the case in the tree of \( \frac{7}{3} \) followed by \( \frac{3}{8} \), whose parents are \( \frac{2}{3} = \frac{1}{3} \) and \( \frac{7}{8} = \frac{1}{2} \). But this example shows how the proof will work, too. We can inductively assume that the claim is true for fractions at the level of \( \frac{a}{b} \) and \( \frac{c}{d} \). In other words \( b = c \). Thus the right child of \( \frac{a}{b} \) has denominator \( b \), and the left child of \( \frac{c}{d} \) has numerator \( c = b \). This the theorem is true in this case, as well. □

Claim 8. The fraction label of node \( n \) has the form \( f(n) / f(n+1) \) for some sequence \( f(n) \).

Proof. This follows immediately from the previous claim. □

Claim 9. The sequence \( f(n) \) is exactly the same as the sequence \( b(n) \) explored earlier.

Proof. What is the left child of node \( n \)? The answer is always \( 2n + 1 \), and this is a fun exercise on its own if there is time. This means the left child of the fraction labelled \( \frac{f(n)}{f(n+1)} \) is the fraction labelled \( \frac{f(2n+1)}{f(2n+2)} \). But then the definition of ‘left child’ tells us that \( f(2n+1) = f(n) \) and \( f(2n) = f(n) + f(n+1) \). Do these statements look familiar? They should, because these are the same statement that defined our hyperbinary sequence \( b(n) \)! □

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