PATTERNS IN DECIMAL EXPANSIONS

1. We know that $1/7 = .\overline{142857}$, and if someone gave us the decimal then we can reconstruct the fraction.

(i) Compute $.\overline{047619}$ and $.\overline{518}$ as fractions. Can you put them in reduced form?

(ii) Compute $.06\overline{81}$ as a reduced fraction. How about $.4\overline{380952}$ as a *reduced* fraction?

2. (i) For each of the following numbers n, find the smallest power of 10 leaving a remainder of 1 when divided by n (i.e., $10^d - 1$ is divisible by n): 7, 13, 37, 41.

(ii) Using the results in (i), express the following as repeating decimals *without* doing further long division: 2/7, 5/13, 1/37, 19/41.

3. (i) Using a calculator, work out the decimal expansions of $1/13, 2/13, \dots, 12/13$. Do the same for $1/11, 2/11, \dots, 10/11$.

(ii) In each of the two cases considered in (i), do you notice any patterns? Can you relate the number of distinct "cycles" of digits in the repeating parts to the number of digits in each such cycle?

(iii) Compute $1/17, 2/17, 3/17, 4/17$ as repeating decimals. Based on the experience in (i) and (ii), can you guess a pattern? Check if it persists for a few more among $5/17, 6/17, \dots$

4. This exercise uses modular arithmetic to explain some classic "divisibility tests".

(i) Using that $10 \equiv 1 \bmod 3$, explain why $5426 \bmod 3$ is the same as $5+4+2+6 = 17$, and that in turn $17 \bmod 3$ is the same as $1+7 = 8$ (so not divisible by 3!). Do the same for 2952. More generally, does this explain a test for divisibility by 3 in terms of the base-10 expression for a whole number?

(ii) Using that $10 \equiv -1 \bmod 11$, explain why $3748 \bmod 11$ is the same as $3(-1)+7+4(-1)+8 = -3+7-4+8 = 8$ (so not divisible by 11!) Do the same for $9416 \bmod 11$. More generally, does this explain a test for divisibility by 11 in terms of the base-10 expression for a whole number?

5. This **bonus** exercise builds on Exercise 4 to develop some exotic "divisibility tests".

(i) Here is a rule for testing if a number $n > 189$ is divisible by 7: subtract twice the units' digit from the part that remains after truncating that digit (i.e., if $n = 10q + u$ with $0 \le u \le 9$ and $q \ge 18$ then pass to $q - 2u < n$). The new smaller number is divisible by 7 precisely when n is.

Explain this by working modulo 7. (Hint: show this is saying that $10q \equiv -u \bmod 7$ precisely when $q \equiv 2u \bmod 7$, and use that $20 \equiv -1 \bmod 7$.) Design a similar test for divisibility by 17, using that $50 \equiv -1 \bmod 17$.

(ii) Here is a rule for testing if a number $n > 49$ is divisible by 13: add 4 times the units' digit to the part that remains after truncating that digit (i.e., if $n = 10q + u$ with $0 \le u \le 9$ and $q \ge 4$ then pass to $q + 4u < n$). The new smaller number is divisible by 13 precisely when n is.

Explain this by working modulo 13. (Hint: show this is saying that $10q \equiv -u \bmod 13$ precisely when $q \equiv -4u \bmod 13$, and use that $40 \equiv 1 \bmod 13$.) Design a similar test for divisibility by 19, using that $20 \equiv 1 \bmod 19$.