2=1+1… And Other Problems Cavemen Could Conquer

Overview		This lesson begins with a deceptively simple idea: decomposing positive integers into the sum of positive integers. As students explore these various compositions, many patterns will emerge. The fun and the challenge lies in explaining why these patterns describe this situation. This lesson will be appropriate for middle school, and is both accessible to all students and extendable for advanced students.
NCTM / Common Core Standards		All 8 Common Core Standards for Mathematical Practice are addressed with this lesson. CCSS.Math.Practice.MP1 Make sense of problems and perservere in solving them. CCSS.Math.Practice.MP2 Reason abstractly and quantitatively. CCSS.Math.Practice.MP3 Construct viable arguments and critique the reasoning of others. CCSS.Math.Practice.MP4 Model with mathematics. CCSS.Math.Practice.MP5 Use appropriate tools strategically. CCSS.Math.Practice.MP6 Attend to precision. CCSS.Math.Practice.MP7 Look for and make use of structure. CCSS.Math.Practice.MP8 Look for and express regularity in repeated reasoning.
Learning Objectives		Students will employ the mathematical practices of systematic thinking, multiple representations, organizing data, and explaining patterns using both recursion and explicit rules. They will also encounter powers of 2 and Pascal's Triangle. Extensions will expose students to combinations.
Materials Required		Pencil and paper (quad-ruled paper may be helpful) Colored pencils, if desired (not necessary)
Instructional Plan	Introduce Problem	Begin by introducing the task to the class as a whole. Illustrate a few ways to decompose a particular number. For example, show that $4=1+2+1$ $\quad = 2+1+1.$ These two compositions should allow students to see that in this activity, order matters. The rule for writing the compositions of a particular integer is that you may write it as **a sum of one or more positive integers**. This bolded phrase will allow students to decide if a particular composition is valid. Allow students to share other compositions, and decide as a class if they are

		valid by using this definition. Once students are comfortable with writing compositions, they are ready to investigate this problem on their own.
	Prompt	"How many compositions does the number 10 have?" Teacher Note: This simple question is enough to ignite students' curiosity, but there are far too many compositions for a student to fully generate. This prompt is a nice entry point, but given the magnitude of this list, do not allow them to struggle for too long at this task. I choose to allow 3-5 minutes of exploration at this point.
	Whole Group Check-in #1	Goal #1: Reinforce estimation as a problem-solving tool. After students have worked for 3-5 minutes, ask them to write down their best guess for the number of compositions that 10 has. Ask them to also write a number they are sure that it exceeds, and a number they are sure it is less than. Take a quick poll to determine the highest guess made and the lowest guess made. Goal #2: Help students determine the need for starting with a smaller case. If many students estimate that 10 has 100 or more compositions, this may be a good time to lead students toward the strategy of starting with a smaller case in the next part of the investigation. They may then return to their investigations. As students work, encourage those who do not begin with decomposing the number 1 to do so, but only if they have had the opportunity to arive at that strategy on their own first.
	Work Time / Student Support	Allot 15 minutes for students to generate data. As students work, you may need to remind them to be organized and methodical. Depending on your students' needs, you may also need to help them record their findings in a table. I choose to intervene only when necessary, without providing too much scaffolding before the need arises. Students working in groups can be tempted to assign each member a different number to decompose, but this could impede their ability to observe patterns. At this point in the problem-solving process, it would be more effective to have all students write the compositions of all numbers, and simply confer with one another to check. As students work, see "Questions for Students" below to find ways to promote deep thinking and finding mathematical connections. At this time, refrain from discussing how students choose to organize the data.

	Whole Group Check-in #2	Goal #1: Check for progress. *"Let's focus on the number 4 since it is not too big, but not too small. Let's begin by collecting the compositions of 4."* Get student responses for the various compositions, and write on the board. *"How do we **know** that we have all of the possibilities?"* This question's intent is to help students articulate how they are generating compositions systematically. If a student can explain her system sufficiently, she can convince herself and others that she has found all compositions. Instruct students to discuss with their peers how they are sure they've found all compositions of 4. After they have had a few minutes, bring the class back together for students to share their way of knowing. This should lead to a systematic organizational scheme that everyone agrees will produce a complete list. Teacher note: The scheme that I am referring to is shown in the first table of the solution pages. It can be described as organizing according to the first number. Goal #2: Introduce students to an alternative representation of the problem. Draw a 1x4 rectangle next to a few of the compositions of 4, and ask students how we could show each composition with a picture. Help students arrive at the conclusion that they could color the number of boxes corresponding to each number in the composition using alternating colors, or shaded and non-shaded boxes. Students may choose to use this visual representation in their investigations. Goal #3: Guide students to use one of the two most accessible organizational schemes. *"Now, with the data organized according to the first number, you can be sure that you have found all compositions of each number. In order to more easily manage these lists, make connections, and find patterns, we may consider organizing each list differently once we generate it systematically."* Prompt students to make suggestions of organizational schemes. Decide which scheme the class will use, or each group will use. Teacher Note: The two most accessible organizational schemes are to organize compositions according to the first number chosen (which is the same as organizing by the last number due to symmetry), and according to the number of terms used. Other organizational schemes are included in the Extensions section of this document. *"During the next work phase, focus on finding and recording data according to your chosen organizational scheme. Begin to hunt for patterns. If a pattern seems to be present, decide if it will continue forever. Explore why this pattern accompanies this problem."*
	Work time / Student	Allot 20 minutes for students to work. Support groups or individuals using the Questions for Students below.

	Support	
	Debrief / Wrap-Up	Bring the class together to debrief and share out. Lead a discussion of why each pattern observed holds true for this problem. See explanations on the solution pages, if desired. In the debrief, I suggest that you explicitly refer to, name, and record all of the problem-solving strategies that were effective in this investigation. These strategies were typed in green throughout the instructional plan above. Highlight for the students that we could only have mastered this problem by organizing our lists. Also, different organizational schemes led to different discoveries. Organization is a very important skill for problem-solving.
Questions for Students		1. As students become confident that they've listed all of a number's compositions, ask each student how he can be sure that he has generated all possibilities. This will promote conversation about each student's scheme for generating a complete list of compositions. 2. As students record their findings, ask how they are keeping track. Guide students who need help creating a chart. Help students to focus on the number of total compositions within each part of their chart. 3. Once a student sees a pattern, ask her to think about whether that pattern will always continue. How can she be sure? This question is intended to bring about discussion of what is causing each pattern to persist. Once that is explained, one can argue that it will continue forever. 4. There are two patterns to notice and explain within the table organized by first number. (See solutions for these.) Any student using this organizational scheme should be prompted to find and explain both. 5. Any student who finds the patterns and describes their underlying cause for one organizational scheme can be prompted to do the same for the other organizational scheme.
Assessment Options		This problem lends itself to informal, formative assessment while students work together, discussing their ideas. Teachers may find it helpful to briefly take notes regarding students' use of the various problem-solving strategies noted in green above. A more formal assessment option is to assign students to give a written explanation of the patterns observed once the class wrap-up has occurred. This may be a homework assignment, an entry in a learning log, or a warm-up at a later date.
Extensions		There are many patterns to observe and explain that can be seen through organizing the compositions for each number differently. Some organizational schemes that have patterns that are more advanced are to organize compositions by the highest term included, or by the number of 1's used.

	Other extensions include altering the rules of decomposing. What if only odd terms are allowed? What if there are two types of 1's, such as red 1's and blue 1's?
Teacher Reflection	I chose to develop a lesson for this problem because I think the systematic generation of data and effectively organizing those data are immensely important problem-solving practices. I was particularly inspired by this problem's wide variety of organizational schemes and different patterns that can be observed from each. In my classroom, I would be sure to highlight this underlying lesson. Additionally, I would remind students of this problem as they encounter more opportunities in future lessons to collect and organize data. On a more practical note, I attempted to use this lesson in my heterogeneous 7th grade classroom. What surprised me most was that the students had more trouble than I anticipated developing a systematic way to collect the data. They felt that a convincing enough answer to the question, "how do you know you've found all the ways?" was that they couldn't find more, or that it was because their list matched another student's list. Eventually, I had to lead a discussion of a good systematic way to collect data so that we could move on to finding and explaining the doubling patterns.
References	I learned this problem from my friend and mentor, Joshua Zucker. In addition to being deeply involved in Math Circles for both teachers and students, he is the founding director of the Julia Robinson Mathematics Festivals. This problem has also appeared at the Julia Robinson Math Festivals. More information about these festivals can be found at http://juliarobinsonmathfestival.org. Joshua played a large part in helping me address this problem thoroughly and write this lesson plan.

Solutions

Below are compositions of 1-6 generated systematically, which is also organized by the first number.

1

1
1

2

2	1+1
1	**1**

3

3	2+1	1+2 1+1+1
1	**1**	**2**

4

4	3+1	2+2 2+1+1	1+3 1+2+1 1+1+2 1+1+1+1
1	**1**	**2**	**4**

5

5	4+1	3+2 3+1+1	2+3 2+2+1 2+1+2 2+1+1+1	1+4 1+3+1 1+2+2 1+2+1+1 1+1+3 1+1+2+1 1+1+1+2 1+1+1+1+1
1	**1**	**2**	**4**	**8**

6

6	5+1	4+2 4+1+1	3+3 3+2+1 3+1+2 3+1+1+1	2+4 2+3+1 2+2+2 2+2+1+1 2+1+3 2+1+2+1 2+1+1+2 2+1+1+1+1	1+5 1+4+1 1+3+2 1+3+1+1 1+2+3 1+2+2+1 1+2+1+2 1+2+1+1+1 1+1+4 1+1+3+1 1+1+2+2 1+1+2+1+1 1+1+1+3 1+1+1+2+1 1+1+1+1+2 1+1+1+1+1+1
1	**1**	**2**	**4**	**8**	**16**

How many compositions does 10 have?
In the table above, the total number of compositions grows in accordance with the powers of 2. The number n has 2^{n-1} compositions. This means that the answer to the original prompt is that 10 has 2^9 compositions.

Explaining the doubling pattern of the total number of compositions:
The first pattern most students will observe is the doubling of the total number of compositions. This pattern is explained by recursion. For instance, for all of the compositions of 5 that begin with 1, the sum of the remaining terms is 4, which can be decomposed in accordance with all of the methods from 4's decomposition. That accounts for 8 compositions beginning with 1. For all compositions beginning with 2, since the sum of the remaining terms is 3, all of the decompositions of 3 will be accounted for there. That adds 4 more compositions to the total. This recursive pattern explains how many compositions will begin with the numbers 1 through n-1. These will be in accordance with descending powers of 2 ranging from 2^{n-2} to 2^0. One more composition is then added to the total to account for the composition in the left-most column: the number itself. Since $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$, this additional composition brings the sum total of compositions of n to 2^n.

Explaining the doubling pattern within the compositions of a particular number based on the value of the first term:
Another doubling pattern becomes apparent for the number of compositions a number has within each option for the first term. The number of compositions doubles when the first term's value decreases by 1. One exception is consistent, which is that there is 1 composition comprised of the number itself, and 1 composition beginning with a term one smaller than the number. After that though, the pattern will hold. This pattern can be explained by focusing on the visual representation of a number. For example, in the case of 5, if the composition begins with 4, you are "gluing" the first 4 blocks together, which you illustrate by coloring them one color. The 5^{th} block must be the opposite color, because otherwise, you would not actually be starting with 4. Therefore, there must be a "break" between the 4^{th} block and the 5^{th} block, and this represents the only option. If the composition begins with 3, you are "gluing" together the first 3 blocks, followed by a break, and the seam between the two blocks remaining could either be "glued" or "broken". Therefore, there are 2 ways to begin with a 3. If you begin with a 2, followed by a break, there will be 2 seams between the remaining 3 blocks that can either be glued or broken. This explains why there are $2 \cdot 2$ or 4 ways to begin with 2. Lastly, by beginning with a 1, followed by a break, there are 3 seams with 2 possibilites each. Therefore, there are 2^3 or 8 compositions beginning with 1. The pattern can be summarized by the statement that as the first term decreases by 1, the number of seams increases by one, and therefore, the number of subsequent possibilities doubles. Again, this pattern has an exception illustrated at the left side of each table, with repeated 1's before the doubling begins.

Below are compositions of 1-6 organized by the number of terms.

1	1
	1

2	2	1+1
	1	**1**

3	3	2+1 1+2	1+1+1
	1	**2**	**1**

4	4	3+1 2+2 1+3	2+1+1 1+1+2 1+2+1	1+1+1+1
	1	**3**	**3**	**1**

5	5	4+1 3+2 2+3 1+4	3+1+1 2+1+2 2+2+1 1+2+2 1+3+1 1+1+3	2+1+1+1 1+1+1+2 1+2+1+1 1+1+2+1	1+1+1+1+1
	1	**4**	**6**	**4**	**1**

6	6	5+1 4+2 3+3 2+4 1+5	4+1+1 3+1+2 3+2+1 2+2+2 2+3+1 1+3+2 1+4+1 2+1+3 1+2+3 1+1+4	3+1+1+1 2+1+1+2 2+2+1+1 1+2+1+2 1+3+1+1 1+1+2+2 2+1+2+1 1+1+1+3 1+2+2+1 1+1+3+1	2+1+1+1+1 1+1+1+1+2 1+2+1+1+1 1+1+2+1+1 1+1+1+2+1	1+1+1+1+1+1
	1	**5**	**10**	**10**	**5**	**1**

<u>Explaining the Pascal's Triangle pattern when organized by number of terms:</u>

For every row, the first and last number of compositions is always 1. This refers to the one way to use only one term (decomposing a number into itself) and the one way to use the maximum number of terms (decomposing into all 1's). This explains the 1's on all of the edges of Pascal's Triangle. The larger, interior numbers come about by adding the two numbers above any particular cell. This is due to the fact that if you are generating all 3-term compositions for the number 6, for example, you can build your compositions from your findings for the number 5. One consideration for using 3 terms to decompose 6 would be to add an extra 1 onto all of the 2-term compositions of 5. The other way to create 3-term compositions of 6 would be to "melt" an extra 1 into the last position of all 3-term compositions of 5. This explains why the number of compositions from the two cells above should be added to give the number of compositions in a particular cell.