The *Futurama* Theorem[1]

adapted for the Philadelphia Math Teacher's Circle
by Cheryl Grood

Appetizer:

Background: This problem (and its solution) appeared in "The Prisoner of Benda," which is episode 7.10 of the television show *Futurama.* The episode's writer, Ken Keeler, has a Ph.D. in applied mathematics from Harvard. (In fact, *several* of the writers on Futurama hold advanced degrees in mathematics, physics, and computer science!) Keeler discovered/proved this theorem entirely as a means to get the characters out of the complicated situation he had written them into. Ken Keeler won the Writers Guild of America Best Television Writing in Animation award in 2010 for this episode.

The setup: Two characters (Professor Farnsworth and Amy) decide to try out their newly-finished "Mind-Switcher" invention on themselves. ("We're just the people this Mind-Switcher was made for by us!") A brief montage of hilarity ensues. When they try to switch back, they discover a key flaw in the machine's design: it will not allow the same pair of bodies to be used in the machine more than once. [clip from show, time stamp 3:33 → 3:55]

The fundamental problem: Is there a way (given the limitations of the machine; *ie*, no cheating by inventing a better machine!) to restore their minds back to their original bodies? The Amy/Professor pair can never re-enter the machine together, but what if they were able to get one or more unsuspecting stooges to enter the machine?

> ***Question:*** Can all correct mind-body pairings be restored with the help of just one other character?
>
> ***Solution:*** No. One new character will buy them at most two more mind-body swaps; brute force exhaustion of the resulting cases reveals no sequence of these swaps will untangle everyone.

---

[1] aka, Keeler's Theorem.

<u>Main Course:</u>

<u>The general problem</u>: [clip from show, time stamp: 3:55 → 4:39]

Is it possible to get everyone back to normal using 4 or more bodies?

<u>Some related questions:</u>

1) **How bad can it get?** Suppose you start with *n* people. Which permutations of minds/bodies are possible? All of them? If not, which ones are unachievable?
2) Can you get everyone back to normal no matter how the minds/bodies are scrambled?
3) When you can restore order:
   a) What is the fewest number of extra bodies needed?
   b) What is the fewest number of swaps needed?

<u>Solution to general problem</u>: Clip from show, time stamp 20:04 → 20:28, informs us:

(Futurama) Theorem: *No matter how a group of people have had their minds and bodies jumbled, it is always possible to restore each person's mind back to its original body, using at most two extra people.*

A sketch of the proof is visible on blackboard in clip, and it is a proof by algorithm—that is, the proof demonstrates a method for restoring order to an arbitrary scrambling. Note that the Futurama Theorem answers more than the general problem asks!

Dessert:

Demonstration of Futurama Theorem unscrambling algorithm:

1) Select two “helper students”. Have remaining students write names on index cards; shuffle these and redistribute randomly. (Explain that this is a way of simulating the results of a hypothetical sequence of swaps via the Mind-Switcher machine. The name on the card you are holding corresponds to the mind that is in your body post-swapping shenanigans.)
2) Give the two “helper students” index cards labeled “Helper 1” and “Helper 2”. They represent people whose minds and bodies have been heretofore unsullied by the Mind-Switcher machine.
3) Get a show of hands from the non-helper students: who received a card with his/her own name? Have those people (if there are any) stand up and move off to the side.
4) Ask for volunteer from remaining seated non-helpers. Have that person stand and read the name on the index card s/he is holding. Whoever’s name is read should get up and stand in front of volunteer.
   a) Now have most recent person who has stood up read name on card, prompting a new person to stand in front of the card-reader.
   b) Repeat (4a) until the name read matches someone already standing (necessarily, this person will be at the back of the line just created).
5) Repeat (4) until the non-helpers have been partitioned into one group of “fixed points” (people holding a card with own name) and one or more lines.
6) Pick one line. Have Helper 1 switch cards with the first person in line.
7) Have Helper 2 switch cards, one at a time, with every person in line, starting at the back of the line. Note at the end of all of these switches, everyone except the helpers and the person at the front of the line is holding a card with his/her own name.
8) Helper 1 switches cards with person in back of line. Now the whole line has correct cards matched with person.
9) Helper 1 & Helper 2 move on to the next line, repeating the procedure in (6)&(7)&(8).
10) After all the lines are fixed, if Helper 1 and Helper 2 have their own cards, we are done; otherwise have them switch.
11) Note that in performing (6)-(10), the same two people never swapped cards. (Why?) Also, even though we don’t know exactly which swaps caused the mess of minds and bodies we started with in (1), we can be confident that none of the swaps in (6)-(10) repeated any of these. (Why?)

Solutions to related questions:

1. Any permutation of *n* objects can be generated by a sequence of non-repeating swaps.
2. Yes, that's the upshot of the Futurama Theorem.
3. (a): This is an open question, with partial results established in the articles by Evans *et al.* The Futurama Theorem shows that an upper bound on the number of additional people is 2; in some cases, such as untangling one swap, fewer than 2 people are needed.
   (b): This is also an open question, with partial results established in the articles by Evans *et al.*

Connections to Curriculum:

- Each mind-switch can be thought of as a function; sequences of swaps correspond to composition of functions; solving the problem amounts to constructing an inverse function using legitimate moves/swaps (F.IF.1, F.IF.2, F.IF.9, F.BF.1.a, F.BF.1.c, F.BF.4.b, F.BF.4.c)
- Enumerating permutations/combinations (S.CP.9)
- Standards for Mathematical Practice!

Bibliography

R. Evans, L. Huang (2012). "Mind Switches in Futurama and Stargate". arXiv:1209.4991v2.

R. Evans, L. Huang, T. Nguyen. Keeler's theorem and products of distinct transpositions, *Amer. Math. Monthly,* 121: 136-144, 2014.

S. Singh, *The Simpsons and Their Mathematical Secrets*, Bloomsbury, New York, 2013.