You can't spell "primes" without "pi"

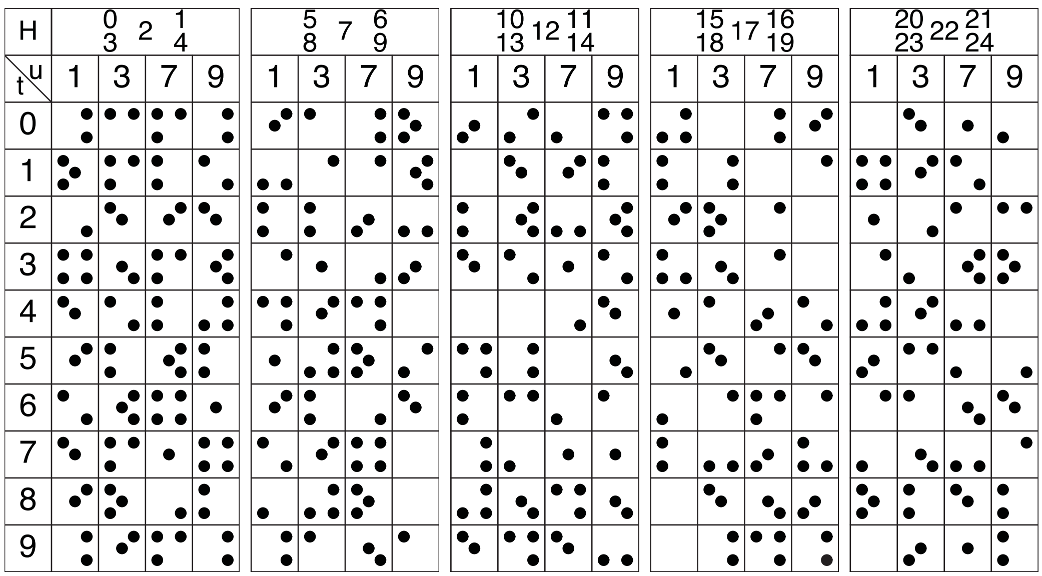
David Patrick
Art of Problem Solving

March 2017

A *prime number* is a positive integer with exactly two positive divisors: 1 and the number itself.

1. Here is a chart of all the prime numbers less than 2500, courtesy of Paul Zeitz:

Prime Numbers to 2500 (+2,5)



http://www.mathteacherscircle.org/assets/session-materials/primecard.pdf

   How do we read this chart? For example, can you use the chart to tell if 1651 is prime or composite? Why does it say "(+2, 5)" at the top? What patterns do you notice? Can you prove them?

2. (a) How many primes are there? (Not just in the chart above, but in the universe.) Can you prove your answer?

   (b) Is there a more descriptive way to ask the question in part (a)? How "dense" are the primes?

3. Why is 1 not a prime? After all, its only positive divisors are 1 and itself: shouldn't that make it prime? What "goes wrong" if we say that 1 is a prime?

4. What's the longest group of consecutive composite numbers? To ask this another way: do the sizes of the gaps between primes have an upper bound, or do these gaps grow without bound?

David Patrick
Art of Problem Solving

5. (a) Are there three consecutive odd numbers that are prime? How often does this occur?
   (b) Are there two consecutive odd numbers that are prime? How often does this occur?

6. If we pick two positive integers at random (what does that even mean?), what's the probability that they're relatively prime?

The *abc* Conjecture

The *abc* Conjecture concerns positive integer solutions to the highly complicated equation

$$a + b = c.$$

If a, b, and c are a solution, and they have a common factor, then we can divide out by that common factor and still have a solution. So, we are only concerned with **primitive** solutions in which a, b, and c have no prime factors in common. From now on, when we say "solution," we mean "primitive solution."

7. Define the **radical** of a solution to be the product of all of the prime factors of a, b, and c. We'll denote this $\mathrm{rad}(abc)$. For example, if $a = 5$ and $b = 7$, then $c = 12$, and $\mathrm{rad}(abc) = 5 \cdot 7 \cdot 2 \cdot 3 = 210$. Experiment by picking a few "random" values of a and b without a common factor, set $c = a+b$, and compute $\mathrm{rad}(abc)$. Do you notice anything interesting?

8. Can you find a solution such that $c = \mathrm{rad}(abc)$? How many such solutions are there?

9. There are exactly 6 solutions with $c < 100$ and $c > \mathrm{rad}(abc)$. Can you find them all? Do you notice any patterns among these solutions?

10. Can you find an infinite number of solutions with $c > \mathrm{rad}(abc)$?

11. Can you find a solution with $c > (\mathrm{rad}(abc))^2$?

12. If you had to try to formulate a conjecture, what would it be?