More Parity problems

(1) Can a 5×5 board be cut into 1×2 dominoes?

(2) John and Pete have three pieces of paper. Each of the boys picks one piece, tears it up, and puts the smaller pieces back. John only tears a piece of paper into 3 smaller pieces while Pete only tears a piece of paper into 5 smaller pieces. After a few minutes can there be exactly 100 pieces of paper?

(3) In a 6×6 chart all but one corner black square are painted white. You are allowed to repaint any column or any row in the chart (i.e., you can select any row or column and change color of all squares within that line). Is it possible to attain an entirely white chart by using only the permitted operations?

(4) Four integers a, b, c, and d produce 6 pairwise sums 2, 4, 9, 9, 14, 16. Is that possible? If a, b, c, and d are not necessarily integers then what are their values?

(5) There are 100 soldiers, and every evening three of them are on duty. Is it possible that after a certain number of days each soldier was on duty with every other soldier exactly once?

(6) All natural numbers from 1 to 101 are written in a row. Can the signs $+$ and $-$ be placed between them so that the value of the resulting expression is 0?

(7) A proper n-sided die is one whose faces are whole numbers between 1 and n (repeats allowed) and whose sum of faces is $= 1 + 2 + \cdots + n = n(n+1)/2$. Let $Pr(n)$ be the number of n-sided proper dice. For example, the 4-sided proper dice are (1,1,4,4), (1,2,3,4), (1,3,3,3), (2,2,2,4), and (2,2,3,3) so $Pr(4) = 5$. Prove that $Pr(n)$ is odd if and only if n is a power of 2.

(8) Prove that among any five points selected inside an equilateral triangle with side equal to 1, there always exists a pair at the distance not greater than .5.

(9) Thirteen line segments are connected end-to-end, so that they form a path. Is it possible that each of these segments crosses exactly one of the other segments?

(10) Can there be a magic square made of the first 36 primes? A magic square here means a 6×6 chart of numbers, so that the sum of numbers along any column, row, or diagonal is the same.

(11) John and Pete have three pieces of paper. Each of the boys picks one piece, tears it up, and puts the smaller pieces back. John only tears a piece of paper into 3 smaller pieces while Pete only tears a piece of paper into 5 smaller pieces. After a few minutes can there be exactly 100 pieces of paper?

(12) There are 100 soldiers, and every evening three of them are on duty. Is it possible that after a certain number of days each soldier was on duty with every other soldier exactly once?

(13) All natural numbers from 1 to 101 are written in a row. Can the signs $+$ and $-$ be placed between them so that the value of the resulting expression is 0?

More Pigeon-hole problems

(1) I have 10 black socks and 10 dark blue socks in a drawer. How many socks do I need to pull out to guarantee that I have a pair of socks to put on?
(2) Given any 6 integers from 1 to 10, some two of them have an odd sum.
(3) If more than half of the integers from 1, 2, ..., 2n are selected, then some two of the selected integers are mutually prime.
(4) Prove that there exists a power of three that ends with 001.
(5) There is a circle of 49 lightbulbs all turned on. Select one bulb as the starting point and repeatedly do the following procedure: If the light bulb you are focusing on is off, then move your attention to the lightbulb to your immediate right. If the lightbulb you are focusing on is on, then reach over to the lightbulb to your immediate right and flip the switch (i.e. turn it off if it's on and on if it's off) and then move your attention to that lightbulb (i.e. the one on your immediate right). Continue moving round and round the circle in this way. Prove that eventually all of the lightbulbs will be on again.
(6) Twenty five boys and twenty five girls sit around a table. Prove that it is always possible to find a person both of whose neighbors are girls.
(7) A person takes at least one aspirin a day for 30 days. If he takes 45 aspirin altogether, in some sequence of consecutive days he takes exactly 14 aspirin.
(8) My wife and I recently attended a party at which there were four other married couples. Various handshakes took place. No one shook hands with oneself, nor with one's spouse, and no one shook hands with the same person more than once. After all the handshakes were over, I asked each person, including my wife, how many hands he (or she) had shaken. To my surprise each gave a different answer. How many hands did my wife shake?
(9) Given 14 or more integers from 1, 2, ... ,28 there exist four of the given integers which can be split into two groups of two with the same sum.
(10) Prove that among any five points selected inside an equilateral triangle with side equal to 1, there always exists a pair at the distance not greater than .5.
(11) Given a planar set of 25 points such that among any three of them there exists a pair at the distance less than 1. Prove that there exists a circle of radius 1 that contains at least 13 of the given points.
(12) In a 14 by 14 grid of dots 58 of the dots are colored blue. Show that there is a rectangle all of whose vertices are blue.
(13) Given any sequence of 100 integers some consecutive subsequence has the property that its sum is a multiple of 100.
(14) Given any sequence of 780 real numbers, some subsequence of 20 numbers is increasing or some subsequence of 42 numbers is decreasing.