Problems for Austin Teachers Math Circle, November 20, 2014
Hats and More!
now with solutions

1. Three prisoners will be given red or blue hats (chosen at random), and arranged so that they can see the other two prisoners' hats, but not their own. They all must simultaneously guess their own hat color. If all three get the answer right, they will be freed. If even one of them gets his color wrong, something horrible will happen to them. (You can supply the details.)

   They are allowed to discuss the problem ahead of time and make a strategy, but there is no possible communication once the hats are revealed. What strategy maximizes their chance of freedom, and what is that probability?

   [If you get stuck, try the analogous problem with 2 prisoners]

   If all three people adopt the following strategy, then there will be a 50% chance of all three being right and all three being wrong: If you see 2 red or no red, guess red. I you see one red, guess blue. This works if the hats are all red or if there are two blues and a red. Another strategy that works is the opposite: if you see 2 red or no red, guess blue. If you see one red, guess red.

2. Same problem, only with N prisoners instead of just 3.

   Everybody should guess that the total number of reds is even (or odd works just as well). Either they are all right (50% chance) or all wrong (50% chance). If they are right, then each person can determine his own color from what he sees: If he sees an odd number of reds, he must be red, and if he sees an even number he must be blue.

3. Now you have N prisoners, arranged in a line. Each prisoner can see the hats of the prisoners in front of him, but not his own or the ones behind. Starting at the back, each prisoner guesses his hat color. If he gets it right, he is freed. If not, he is punished (again, supply the gory details yourself). As before, the prisoners can decide on their strategy ahead of time. What strategy will minimize the average number of prisoners who get punished? [You may want to start with $N = 2$ and work up.]

The person in back says "red" if he sees an odd number of reds and "blue" if he sees an even number of reds. This provides a "check bit" for all of the other people. Everybody sees all the people in front, and hears all the people in back. From the check bit he can determine his own color. The end result is that the person in back has a 50-50 chance, and everybody else will go free.

4. Same setup of problem 1, except that a) the prisoners don't have to guess simultaneously (but they do have to guess within, say, one hour), b) when each person guesses, everybody else gets to hear the guess, and c) everybody is freed if ALL the guesses are right, but everybody is punished if even one person gets it wrong.

   Find a strategy with a high probability of getting everybody released.

   There are two pieces to the strategy, depending on whether anybody else has guessed yet or not.

   If nobody else has spoken yet, wait $\min(r, b)$ *minutes, where* r *is the number of red hats you see and* b *is the number of blue hats you see. Then guess red if* $r < b$ *and blue if* $b < r$*. You can never have* $r = b$ *in this situation, since in that case somebody else would have a smaller value of* r *or a smaller value of* b*, and would have spoken up already.*

   If somebody has spoken up already, then you must have a different hat color from the guy who spoke up. (Since if you had the same color you would have seen the same thing that he saw, and would have spoken up at the same time that he did.)

   For instance, if there are 5 red hats and 8 blue hats, then all of the reds see an 8-4 split and say "red" after 4 minutes. All of the blues see a 7-5 split, don't say anything at the 4 minute mark, and realize that they must be blue.

   The only situation in which this fails is if everybody has the same hat color, in which case everybody blurts out the wrong color immediately. This has probability 2^{1-N}*. For* $N = 18$*, that's less than one chance in 100,000.*

5. And now for something completely different. Let B be a convex body in the plane. Show that there is a way to cut B along a straight line so that each piece has the same area and the same perimeter.

For each direction θ, you can find a line pointing in the correct direction that divides the perimeter in half. Let $f(\theta)$ be the area on the left minus the area on the right. Note that $f(\theta + \pi) = -f(\theta)$, since those are the same line, only pointing in opposite directions. But f is continuous, so if $f(\theta)$ is non-zero, there must be an angle α between θ and $\theta + \pi$ such that $f(\alpha) = 0$. (You could also divide the area in half and track the difference in perimeter between the two halves. The argument is essentially the same.)

6. And another completely different problem. Show that some number of the form 99999. . .9 is divisible by 324,137. (And no, there is nothing special about the number 324,137.)

   I slightly lied. There is something a little bit special about 324,137, namely that it is an odd number that isn't a multiple of 5.

   There are infinitely many integers of the form $10^n - 1 = 99999\ldots9$, and only finitely many possible remainders when you divide by $N = 324,137$. By the pigeonhole principle, this means that two such numbers $10^n - 1$ and $10^m - 1$ (say, with $n > m$) must have the same remainder, and hence that the difference between these numbers is divisible by N. But this difference is of the form $9\ldots90\ldots0 = 10^m(10^{n-m} - 1)$ But N and 10^m have no common factors, so N must divide $10^{n-m} - 1$.