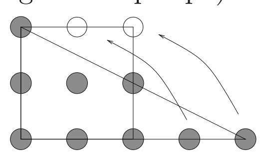
Follow-up on geoboards, ways of making change, and Pick's theorem

Thanks to everyone for a very interesting MTC! I learned a lot from the issues that came up.

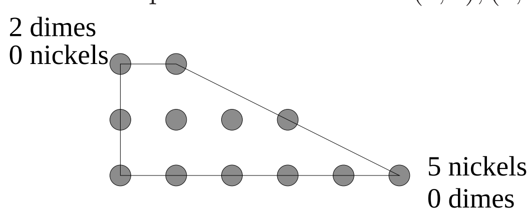
(i) Geoboards. We didn't have time to demonstrate the online tool, but it is free and easily available. So there are activities one could do with polygons, without having geoboards available in the classroom. (Just google online geoboard, or try http://www.mathlearningcenter.org/web-apps/geoboard/)

(ii) Number of ways of making change for a dollar. This is a problem that a lot of fourth graders get sent home with, and we tried to make the connection with geometry here. We talked about how the number of ways of making change for a multiple of ten cents using pennies, nickels and dimes is the number of lattice points in a lattice triangle whose hypotenuse has slope $-1/2$. The number of such points is always a square (you can rearrange the points in the triangle into the points in a square, by taking the points on the "pointy half" and moving them up top"):



So for example, the number of ways of making change for 50 cents is 36, and for a dollar, 121, without using quarters; for making change for $10n$ cents there are $(n+1)^2$ ways.

The number of ways of making change for a number like 25, 75 using pennies, nickels and dimes is the number of lattice points inside a trapezoid. (For 25 cents, this is the trapezoid with vertices $(0,0), (0,2), (1,2), (5,0)$.)



Cutting off the left bit of the trapezoid one gets a triangle. The number of ways of making change for 25 cents is $3 + 9 = 12$ ways, and the number of ways of making change for 75 cents is $8 + 64 = 72$ ways. (It's easier to see this if you draw the trapezoid here, with vertices $(0,0), (0,7), (1,7), (15,0)$.)

To make change for a dollar, you can either use 0,1,2,3, or 4 quarters. The leftover amounts are 0,25,50,75,100 cents which you have to make change for using pennies, nickels and dimes. So in total there are $1 + 12 + 36 + 72 + 121 = 242$ ways of making change for a dollar (trapezoid + triangle + trapezoid + triangle) without using half-dollar or dollar coins.

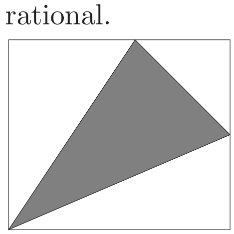
If you allow one half-dollar, then you get another $1 + 12 + 36 = 49$ ways, and one can also use two-half dollars or just a dollar coin(does this count as making change?) for a total of 293 ways. There are a bunch of discussions of this on the web, see e.g. http://www.maa.org/frank-morgans-math-chat-293-ways-to-make-change-for-a-dollar.

(i) Pick's theorem. A key issue that came up, as Ann said, is data collection: this was a key feature in how quickly people discovered Pick's theorem (which says that the number of lattice points inside or on the boundary of a lattice polygon is the area, plus half the number of lattice points on the boundary, plus one; there is a slightly different formula if one only counts lattice points strictly inside.) I think the key point here is that one has to mess around a little bit first, in order to set up a data collection scheme, and then go back and do things more systematically; the data collection scheme has to be "designed" based on some experience.

Another issue that came up is what kind of formula can one expect? I think this would have been a good question to ask people to guess at the beginning. Could one expect a formula just expressing the area in terms of the number of lattice points? why or why not? etc.

We didn't have time to talk about how to show that the formula works for all shapes. One argument is to look at the quantity $n(P)$ given by "number of points inside or on the boundary minus half the points on the boundary minus one" and show that it satisfies an "additivity principle": whenever P is divided up into P_1 and P_2 then $n(P) = n(P_1) + n(P_2)$. For triangles of area one-half, one shows that the $n(P)$ is $1/2$, which is the same as the area. But any lattice polygon can be divided into lattice triangles of area one-half (challenge)! Another explanation is at http://www.jamestanton.com/wp-content/uploads/2009/04/picks_theorem_focus_web-version.pdf

(iv) Equilateral triangles on geoboard. Equilateral triangles cannot be made on a geoboard. Here is one way of seeing this (Maybe there is an easier way, I'm not sure.) First, prove that any triangle drawn on a geoboard must have a rational area. To that end, you can draw vertical lines through the left-most and right-most vertices of the triangle, and horizontal lines through the highest and lowest vertices. They will intersect to form a rectangle whose sides are of integer length, so its area is an integer. The area of the triangle inside is the area of this rectangle minus the sum of the areas of 3 little triangles that are inside the rectangle but outside the initial triangle. Integer minus rational gives rational.



Now, the formula for the area of an equilateral triangle is $side^2 * \sqrt{3}/4$. Because the coordinates of the triangle's vertices are integers, $side^2$ is an integer as well. So, $side^2 * \sqrt{3}/4$ cannot be rational, therefore this triangle does not exist.